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The complexity of approximately counting stable matchings ☆

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ABSTRACT

We investigate the complexity of *approximately counting* stable matchings in the k-attribute model, where the preference lists are determined by dot products of "preference vectors" with "attribute vectors", or by Euclidean distances between "preference points "and "attribute points". Irving and Leather (1986) [14] proved that counting the number of stable matchings in the general case is #P-complete. Counting the number of stable matchings is reducible to counting the number of downsets in a (related) partial order [14] and is interreducible, in an approximation-preserving sense, to a class of problems that includes counting the number of independent sets in a bipartite graph (#*BIS*) (Dyer et al. (2004) [6]). It is conjectured that no FPRAS exists for this class of problems. We show this approximation-preserving interreducibility remains even in the restricted k-attribute setting when $k \geq 3$ (dot products) or $k \geq 2$ (Euclidean distances). Finally, we show it is easy to count the number of stable matchings in the 1-attribute dot-product setting.

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1. Introduction

1.1. Stable Matchings

The *stable matching problem* (or *stable marriage problem*) is a classical combinatorics problem. An instance of this problem consists of n men and n women, where each man has his own preference list (a total ordering) of the women, and, similarly, each woman has her own preference list of the men. A one-to-one pairing of the men with the women is called a *matching* (or *marriage*). Given a matching, if there exists a man M and a woman w in the matching who prefer each other over their partners in the matching, then the matching is considered *unstable* and the man–woman pair (M, w) is called a *blocking pair*. (M and w would prefer to drop their current partners and pair up with each other.) If a matching has no blocking pairs, then we call it a *stable matching*. In 1962, Gale and Shapley proved that every stable matching instance has a stable matching, and described an $O(n^2)$ algorithm for finding one [7].

The stable matching problem has many variants, where ties in the preference lists could be allowed, where people might have partial preference lists (i.e. someone might prefer to remain single rather than be paired with certain members of the opposite sex), generalizations to men/women/pets, universities and applicants, students and projects, etc. Some of these generalizations have also been well-studied and, indeed, algorithms for finding stable matchings are used for assigning residents to hospitals in Scotland, Canada, and the USA [3,18,20].

In this paper, we concentrate solely on the classical problem, so the term "matching instance" will refer to one where the number of men is equal to the number of women, and each man or women has their own full totally-ordered (i.e. no ties allowed) preference list for the opposite sex.

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Irving and Leather [14] demonstrated that counting the *number* of stable matchings for a given instance is #P-complete. This completeness result relies on the connection between stable marriages and *downsets* in a related partial order (explained in more detail in Section 3), as counting the number of downsets in a partial order is another classical #P-complete problem [19].

Knowing that exactly counting stable matchings is difficult (under standard complexity-theoretic assumptions), one might turn to methods for *approximately* counting this number. In particular, we would like to find a *fully-polynomial randomized approximation scheme* (an *FPRAS*) for this task, i.e. an algorithm that provides an arbitrarily close approximation in time polynomial in the input size and the desired error — see Section 2 for a formal definition. One method that has proven successful for other counting problems is the Markov Chain Monte Carlo (MCMC) method. This technique exploits a relationship between counting and sampling described by Jerrum et al. [15], namely, for *self-reducible* combinatorial structures, the existence of an FPRAS is computationally equivalent to a polynomial-time algorithm for approximate sampling from the set of structures. Although the set of stable matchings for an instance does not obviously fit into the class of self-reducible problems, an efficient algorithm for (approximately) sampling a random stable matching can be transformed into a method for (approximately) counting this number.

Bhatnagar et al. [1] considered this problem of sampling a random stable matching using the MCMC method. They examined a natural Markov chain that uses "male-improving" and "female-improving" *rotations* (see Section 3.3) to define a random walk on the state space of stable matchings for a given instance. In the most general setting, matching instances can be exhibited for which the *mixing time* of the random walk has an exponential lower bound, meaning that it will take an exponential amount of time to (approximately) sample a random stable matching. This exponential mixing time is due to the existence of a "bad cut" in the state space. Bhatnagar, et al. considered several restricted settings for matching instances and were still able to show instances for which such a bad cut exists in the state space, implying an exponential mixing time in these restricted settings.

Of particular interest to us in this paper, Bhatnagar et al. examined the so-called *k-attribute model*. In this setting each man and woman has two k-dimensional vectors associated with them, a "preference" vector and a "position" (or "attribute") vector. A man M_i has a preference vector denoted by $\hat{M_i}$, and a position vector denoted by $\bar{M_i}$ (similarly denoted for the woman w_j). Then, M_i prefers w_j over w_ℓ (i.e. w_j appears higher on his preference list than w_ℓ) if and only if $\hat{M_i} \cdot \bar{w}_j > \hat{M_i} \cdot \bar{w}_\ell$, where $\hat{M_i} \cdot \bar{w}_j$ denotes the usual k-dimensional dot product of vectors. Since we assume that each man has a total order over the women (and vice-versa), we note that $\hat{M_i} \cdot \bar{w}_j \neq \hat{M_i} \cdot \bar{w}_\ell$ whenever $j \neq \ell$ (and analogously for the women's preference vectors/men's position vectors).

Even in this restricted k-attribute setting (not every matching instance can be represented in this manner if k is small [2]), Bhatnagar, Greenberg, and Randall were still able to demonstrate examples of matching instances having a "bad cut" where the Markov chain has an exponential mixing time. Bhatnagar et al. also considered two other restricted settings, the so-called *k-range* and *k-list* models, but we will not be considering those cases here. (Again, they gave instances having an exponential mixing time for the Markov chain.)

It must be noted that even though the male-improving/female-improving Markov chain might have an exponential mixing time, this does not necessarily imply the non-existence of an FPRAS for the corresponding counting problems. However, Dyer et al. [6] give evidence suggesting that even *approximately* counting the number of stable matchings is itself difficult, i.e. suggesting that an FPRAS may not exist. They do this by demonstrating *approximation-preserving reductions* amongst several counting problems, one being that of counting downsets in a partial order (once again, the connection to stable matchings is outlined in Section 3). Relevant background about approximation-preserving reductions is discussed in Section 2. The main point is that the existence of an FPRAS for one problem would imply the existence of an FPRAS for this entire class of counting problems. Currently, the existence of such an FPRAS remains an open question.

It is precisely the goal of this paper to consider the complexity of the approximate counting problem for the k-attribute model.

Before we continue, let us formally define some counting problems. Two counting problems relevant to us are:

Name: #*SM*.

Instance: A stable matching instance with n men and n women.

Output: The number of stable matchings.

Name: #*SM*(k-attribute).

Instance: A stable matching instance with n men and n women in the k-attribute setting, i.e. preference lists are determined using dot products between k-dimensional preference and position vectors as mentioned above.

Output: The number of stable matchings.

As we stated previously, if k is small (relative to n), there exist preference lists that are not realizable in the k-attribute setting [2]. On the other hand, if $k = n$ then we can clearly represent any set of n preference lists by simply using a separate coordinate for each person to rank the members of the opposite sex.

Another counting problem we consider in this paper is the following one:

Name: #*SM*(k-Euclidean).

Instance: A stable matching instance with n men and n women in the k-dimensional Euclidean setting. In this setting, men and women each have a "preference point" and "position point". Preference lists are determined using Euclidean distances between preference points and position points.

Output: The number of stable matchings.

In other words, for a k-Euclidean stable matching instance man M_i prefers woman w_j to woman w_ℓ if and only if $d(\hat{M}_i, \bar{w}_j) < d(\hat{M}_i, \bar{w}_\ell)$, where $d(x, y)$ is the Euclidean distance between points x and y. Once again, ties are not allowed in the preference lists.

Before we describe our results, let us give a brief introduction to approximation-preserving (AP) reductions and AP-reducibility. Further details can be found in Section 2.

1.2. AP-reducibility (a brief introduction)

Approximate counting problems have been of increasing interest in recent years. Some success has been demonstrated by finding fully-polynomial randomized approximation schemes for some $\#P$-complete problems. Likewise, there are some (but fewer) problems known to not admit an FPRAS under usual complexity-theoretic assumptions.

AP-reducibility (for *approximation-preserving reducibility*) is similar in nature to reductions used in showing problems are NP-complete. Broadly speaking, if g is an integer-valued function (that counts some type of combinatorial structure) and there is an approximation-preserving reduction from another integer-valued function f (counting something else) to g, then an FPRAS for g gives us an FPRAS for f. (Similar to Turing reductions, the problem sizes are polynomially related and the error terms of the approximations are also polynomially related.) In this case we would write $f \leq_{AP} g$ to mean that f has an AP-reduction to g. Similarly we write $f \equiv_{AP} g$ to mean that $f \leq_{AP} g$ *and* $g \leq_{AP} f$, or that f and g are AP-interreducible. Definitions are provided in Section 2.

This kind of AP-reduction allows us to study the relative complexity of approximate counting problems, just as polynomial many-one reductions allow us to compare the complexity of decision problems such as graph coloring and satisfiability.

The complexity class $\#\mathrm{RH}\Pi_1$ of counting problems was introduced by Dyer et al. [6] as a means to classify a wide class of approximate counting problems that were previously of indeterminate computational complexity. The problems in $\#\mathrm{RH}\Pi_1$ are those that can be expressed in terms of counting the number of models of a logical formula from a certain syntactically restricted class. Although the authors were not aware of it at the time, this syntactically restricted class had already been studied under the title "restricted Krom SNP" [4]. The complexity class $\#\mathrm{RH}\Pi_1$ has a completeness class (with respect to AP-reductions) which includes a wide and ever-increasing range of natural counting problems, including: independent sets in a bipartite graph, downsets in a partial order, configurations in the Widom-Rowlinson model (all [6]) and the partition function of the ferromagnetic Ising model with mixed external field [9]. Either all of these problems have an FPRAS, or none do. No FPRAS is currently known for any of them, despite much effort having been expended on finding one.

All the problems in the completeness class mentioned above are inter-reducible via AP-reductions, so any of them could be said to exemplify the completeness class. However, mainly for historical reasons, the following problem tends to be taken as a key example in the class, much in the same way that SAT has a privileged status in the theory on NP-completeness.

Name: #*BIS*.

Instance: A bipartite graph B.

Output: The number of independent sets in B.

Ge and Štefankovič [8] recently proposed an interesting new MCMC algorithm for sampling independent sets in bipartite graphs. Unfortunately, however, the relevant Markov chain mixes slowly [11] so even this interesting new idea does not give an FPRAS for #*BIS*. In fact, Goldberg and Jerrum [10] conjecture that no FPRAS exists for #*BIS* (or for the other problems in the completeness class). We make this conjecture on empirical grounds, namely that the problem has survived its first decade despite considerable efforts to find an FPRAS and the collection of known #*BIS*-equivalent problems is growing.

Since Dyer et al. show that #*BIS* and counting downsets are both complete in this class, and it is known that counting downsets is equivalent to counting stable matchings, the result of Dyer et al. implies $\#BIS \equiv_{AP} \#SM$.

The goal of this paper is to demonstrate AP-interreducibility of #*BIS* with the two restricted stable matching problems defined in Section 1.1.

1.3. Our results

In this paper we prove the following results:

Theorem 1. *$\#BIS \equiv_{AP} \#SM$(k-attribute) when $k \geq 3$.*

In other words, #*BIS* is AP-interreducible with counting stable matchings in the k-attribute setting when $k \geq 3$, so this problem is equivalent in terms of approximability to the complete problems in the complexity class $\#\mathrm{RH}\Pi_1$.

Theorem 2. *#SM(1-attribute) is solvable in polynomial time.*

We can also prove AP-interreducibility with #*BIS* in the k-Euclidean setting (when $k \geq 2$) in a similar manner. Recall that in the k-Euclidean setting, preference lists are determined by (closest) Euclidean distances between the "preference points" and "position points".

Theorem 3. *#BIS* $\equiv_{AP}$ *#SM(k-Euclidean) when* $k \geq 2$.

The rest of the paper is laid out as follows:

We review further background on approximation schemes and AP-reductions in Section 2.

Section 3 reviews some combinatorics of the stable matching problem that is relevant for our purposes in this paper.

Section 4 demonstrates Theorem 1 and Section 5 is devoted to proving Theorem 2.

We give the construction required to demonstrate Theorem 3 in Section 6. This construction ends up giving us identical preference lists as those for the k-attribute ($k \geq 3$) model. Thus, the remainder of the proof to show AP-interreducibility between #*SM*(k-Euclidean) for $k \geq 2$ and #*BIS* is identical to that for the 3-attribute setting and is not repeated.

2. Randomized approximation schemes and approximation-preserving reductions

A *randomized approximation scheme* is an algorithm for approximately computing the value of a function $f : \Sigma^* \to \mathbb{R}$. The approximation scheme has a parameter $\varepsilon > 0$ which specifies the error tolerance. A *randomized approximation scheme* for f is a randomized algorithm that takes as input an instance $x \in \Sigma^*$ (e.g., for the problem #*SM*, the input would be an encoding of a stable matching instance) and a rational error tolerance $\varepsilon > 0$, and outputs a rational number z (a random variable of the "coin tosses" made by the algorithm) such that, for every instance x,

$$\Pr\left[e^{-\epsilon} f(x) \leq z \leq e^{\epsilon} f(x)\right] \geq \frac{3}{4}. \tag{1}$$

The randomized approximation scheme is said to be a *fully polynomial randomized approximation scheme*, or *FPRAS*, if it runs in time bounded by a polynomial in $|x|$ and ϵ^{-1}. Note that the quantity 3/4 in Eq. (1) could be changed to any value in the open interval $(\frac{1}{2}, 1)$ without changing the set of problems that have randomized approximation schemes [15, Lemma 6.1].

We now define the notion of an approximation-preserving (AP) reduction. Suppose that f and g are functions from Σ^* to $\mathbb{R}$. As mentioned before, an AP-reduction from f to g gives a way to turn an FPRAS for g into an FPRAS for f. Here is the formal definition. An *approximation-preserving reduction* from f to g is a randomized algorithm $\mathcal{A}$ for computing f using an oracle for g. The algorithm $\mathcal{A}$ takes as input a pair $(x, \varepsilon) \in \Sigma^* \times (0, 1)$, and satisfies the following three conditions: (i) every oracle call made by $\mathcal{A}$ is of the form (w, δ), where $w \in \Sigma^*$ is an instance of g, and $0 < \delta < 1$ is an error bound satisfying $\delta^{-1} \leq \text{poly}(|x|, \varepsilon^{-1})$; (ii) the algorithm $\mathcal{A}$ meets the specification for being a randomized approximation scheme for f (as described above) whenever the oracle meets the specification for being a randomized approximation scheme for g; and (iii) the run-time of $\mathcal{A}$ is polynomial in $|x|$ and ε^{-1}.

According to the definition, approximation-preserving reductions may use randomization and may make multiple oracle calls. Nevertheless, the reductions that we present in this paper are deterministic. Each reduction makes a single oracle call (with $\delta = \epsilon$) and returns the result of that oracle call. A word of warning about terminology: Subsequent to [6], the notation $\leq_{\text{AP}}$ has been used to denote a different type of approximation-preserving reduction which applies to optimization problems. We will not study optimization problems in this paper, so hopefully this will not cause confusion.

3. Combinatorics of the stable matching problem

The (classical) stable matching problem has a rich combinatorial structure which has been widely studied. We relate some aspects of this structure that we will need in this paper. Many of the definitions and results that follow can be found, for example, in [16,14,13,12].

3.1. The Gale–Shapley algorithm

In their seminal paper on the stable matching problem, Gale and Shapley [7] gave a polynomial-time algorithm for constructing a stable matching. This is generally referred to as the "proposal algorithm" and bears the names of Gale and Shapley in all of the literature on stable matchings. One sex (typically the men) make proposals to members of the other, forming "engagements". Once all the "proposers" are engaged, the algorithm terminates with a stable matching.

A description of this algorithm follows.

As noted by Gale and Shapley (and others), the above algorithm computes the male-optimal stable matching, which is optimal in the very strong sense that every man likes his partner in this matching at least as much as his partner in any other stable matching. Given an instance with n men and n women, the algorithm computes the male-optimal stable matching in time $O(n^2)$.

During the algorithm, after a woman becomes "engaged" she never becomes free, though she might be engaged to different men at different times during the execution of the algorithm. On the other hand, a man could oscillate between being free and being engaged.

Algorithm 1 Gale–Shapley Algorithm

- Initially every man and every woman is free.
- Repeat until all men are engaged:
- A free man M proposes to w, the highest woman on his list who has not already rejected him.
  - If w is free, then she accepts the proposal and M and w become engaged.
  - If w is engaged to M', then
    - Let M^+ be the favorite of w between men M and M'.
    - Let M^- be the least favorite of w between men M and M'.
    - M^+ and w become engaged to each other.
    - w rejects M^- and M^- is set free.

It is well-known (see, e.g. [7,16]) that the male-optimal matching may be obtained by taking *any* ordering of the men and have them make proposals in that order, i.e. when "a free man M proposes..." we can take the highest free man in our ordering of the men to perform the next proposal.

By reversing the roles of men and women (i.e. the women are the "proposers"), we can obtain the female-optimal stable matching.

3.2. Stable matching lattice

Given a matching instance and two stable matchings $\mathcal{M}$ and $\mathcal{M}'$ where

$$\mathcal{M} = \{(M_1, w_1), (M_2, w_2), \ldots, (M_n, w_n)\},$$
$$\mathcal{M}' = \{(M_1', w_1), (M_2', w_2), \ldots, (M_n', w_n)\},$$

we define $\max\{M_i, M_i'\}$, $\min\{M_i, M_i'\}$, $\max\{\mathcal{M}, \mathcal{M}'\}$ and $\min\{\mathcal{M}, \mathcal{M}'\}$ as follows:

$$\max\{M_i, M_i'\} = \text{ favorite choice of woman } w_i \text{ between men } M_i \text{ and } M_i'$$
$$\min\{M_i, M_i'\} = \text{ least preferred choice of woman } w_i \text{ between men } M_i \text{ and } M_i'$$
$$\max\{\mathcal{M}, \mathcal{M}'\} = \{(\max\{M_1, M_1'\}, w_1), (\max\{M_2, M_2'\}, w_2), \ldots, (\max\{M_n, M_n'\}, w_n)\}$$
$$\min\{\mathcal{M}, \mathcal{M}'\} = \{(\min\{M_1, M_1'\}, w_1), (\min\{M_2, M_2'\}, w_2), \ldots, (\min\{M_n, M_n'\}, w_n)\}$$

Note that in the expression $\max\{M_i, M_i'\}$, the woman w_i can be deduced from the arguments since she is the only woman married to M_i in $\mathcal{M}$ and to M_i' in $\mathcal{M}'$. From [16], we have that $\max\{\mathcal{M}, \mathcal{M}'\}$ and $\min\{\mathcal{M}, \mathcal{M}'\}$ are themselves stable matchings. Further, we define the relation $\mathcal{M} \leq \mathcal{M}'$ if and only if $\mathcal{M}' = \max\{\mathcal{M}, \mathcal{M}'\}$. It is clear that the relation $\leq$ is reflexive, antisymmetric, and transitive. Hence, the stable matchings of a stable matching instance form a lattice under the $\leq$ relation.

In fact, this lattice is a *distributive lattice* under the "max" and "min" operations defined above [16]. The male-optimal matching is the minimum element in this lattice (under the $\leq$ relation), while the female-optimal matching is the maximum element.

It is well-known (see, for instance, [5]) that a finite distributive lattice is isomorphic to the lattice of *downsets* of another partial order (ordered by subset inclusion). We shall shortly see how this other downset lattice arises in the context of stable matchings, and its connection to the stable matching lattice.

3.3. Stable pairs and rotations

Definition 3.1. A pair (M, w) is called *stable* if and only if (M, w) is a pair in some stable matching $\mathcal{M}$. A pair (M, w) that is not stable is called an *unstable pair*.

Definition 3.2. Let $\mathcal{M}$ be a stable matching. For any man M (woman w), let $sp_{\mathcal{M}}(M)$ ($sp_{\mathcal{M}}(w)$) denote the spouse of man M (woman w) in the matching $\mathcal{M}$.

Definition 3.3 (*[1]*)**.** Let $\mathcal{M}$ be a stable matching. The *suitor* of a man M is defined to be the first woman w on M's preference list such that (i) M prefers his spouse over w and (ii) w prefers M over her spouse. The suitor of man M is denoted by $S_{\mathcal{M}}(M)$.

We note that $S_{\mathcal{M}}(M)$ may not exist for every man. For instance, if $\mathcal{M}$ is the female-optimal stable matching, then $S_{\mathcal{M}}(M)$ would not exist.

Definition 3.4 (*[14]*)**.** Let $\mathcal{M}$ be a stable matching. Let $R = \{(M_0, w_0), (M_1, w_1), \ldots, (M_{k-1}, w_{k-1})\}$ be an ordered list of pairs from $\mathcal{M}$ such that for every i, $0 \leq i \leq k-1$, $S_{\mathcal{M}}(M_i)$ is $w_{i+1(\mod k)}$. Then R is a *rotation* (exposed in the matching $\mathcal{M}$).

A stable matching may have many or no exposed rotations. Applying an exposed rotation to a stable matching (i.e. breaking the pairs (M_i, w_i) and forming the new pairs (M_i, w_{i+1})) gives a new stable matching in which the women are "happier" and the men are less happy. In other words, after a rotation, every woman (respectively, man) involved in the rotation is married to someone higher (resp. lower) on her (resp. his) preference list than her (resp. his) partner in the rotation.

We can similarly define suitors for the women, given some stable matching $\mathcal{M}$. We do not need to do so for the purposes of this paper, but the Markov chain that Bhatnagar, et al. examine in [1] consists of moves that are "male-improving" and "female-improving" rotations. Starting from any stable matching, it is possible to obtain any other stable matching using some (appropriately chosen) sequence of male-improving and/or female-improving rotations [14].

Definition 3.5 (*[12]*)**.** A pair (M, w), not necessarily stable, is said to be *eliminated by the rotation R* if R moves w from M or below on her preference list to a man strictly above M.

Note that if a stable pair (M, w) in a rotation R is eliminated by R, and if (M, w') is any other pair eliminated by R, then man M prefers w over w', for otherwise no matching that has R exposed in it could be stable.

Lemma 4 (*[14]*)**.** *No pair is eliminated by more than one rotation, and for any pair (M, w), at most one rotation moves M to w.*

We can now define a relation on rotations.

Definition 3.6 (*[14]*)**.** Let R and R' be two distinct rotations. Rotation R is said to *explicitly precede* R' if and only if R eliminates a pair (M, w), and R' moves M to a woman w' such that M (strictly) prefers w to w'. The relation "precedes" is defined as the transitive closure of the "explicitly precedes" relation.

If a rotation R explicitly precedes R' then there is no stable matching with R' exposed such that applying R' results in a stable matching with R exposed — the intermediate matching would have a blocking pair (hence would not be stable). The relation *precedes* ($\leq$) defines a partial order on the set of rotations of the stable matching instance. We call the partial order on the set of rotations the *rotation poset* of the instance and denote it $(P, \leq)$.

The following theorem relates the rotations in the rotation poset to the stable matchings of the instance via the downsets of P.

Theorem 5 (*[14, Theorem 4.1]*)**.** *For any stable matching instance, there is a one-to-one correspondence between the stable matchings of that instance and the downsets of its rotation poset.*

Every stable matching of the instance can be obtained by starting with the male-optimal stable matching and performing the rotations in the corresponding downset (ensuring that a rotation is performed before any rotation that succeeds it is performed). Note that the downsets corresponding to the male-optimal stable matching and the female-optimal stable matching are $\emptyset$ and P, respectively.

To construct the rotation poset, we need (i) the rotations and (ii) the precedence relations between them. We note that once we have all the rotations in the poset, we can establish the precedence relations using the "explicitly precedes" relation, i.e. by determining which (stable or unstable) pairs are eliminated by each rotation.

3.4. Gusfield's algorithm for finding all rotations

Given a stable matching instance, let H be the Hasse diagram of the stable matching lattice defined in Section 3.2. That is, H is the transitive reduction of the relation $\leq$ on the set of stable matchings. Gusfield [12] gave a fast algorithm for finding all rotations of a stable matching instance. His algorithm is a refinement of successive applications of the "*breakmarriage*" procedure of McVitie and Wilson [17]. The key ingredient in Gusfield's proof that his algorithm is correct is the following.

Theorem 6 (*[12, Theorem 6]*)**.** *Let Φ be any path in H from the male-optimal stable matching to the female-optimal stable matching. Then any two consecutive matchings on Φ differ by a single rotation, and the set of all rotations between matchings along Φ contains every rotation exactly once.*

Gusfield presented a well-tuned version of his algorithm that runs in $O(n^2)$ time. For the sake of presentation, we use the following, slower, variant of his algorithm (the variant still runs in polynomial time, which suffices for our purposes).

In Algorithm 2, the existence of t is guaranteed by the fact that the current stable matching $\mathcal{M}_i$ is different from the female-optimal stable matching and, hence, has a rotation exposed in it. The correctness of the algorithm follows from Theorem 6: Starting at $\mathcal{M}_i$, the algorithm applies a rotation to obtain $\mathcal{M}_{i+1}$. Since rotations improve the utility of the women involved, $\mathcal{M}_{i+1} \geq \mathcal{M}_i$. To apply Theorem 6 we need only argue that the step from $\mathcal{M}_i$ to $\mathcal{M}_{i+1}$ is a single step in H rather than multiple steps. Equivalently, we need to argue that the rotation between $\mathcal{M}_i$ and $\mathcal{M}_{i+1}$ cannot be decomposed as a sequence of smaller rotations (where these smaller rotations correspond to individual steps in H). This follows by the definition of "suitor" — if $\mathcal{M}_i$ has rotation R exposed and applying rotation R yields a stable matching $\mathcal{M}'$ with R' exposed and applying R' yields $\mathcal{M}_{i+1}$ then either R and R' are disjoint (contradicting the fact that the transformation from $\mathcal{M}_i$ to $\mathcal{M}_{i+1}$ can be accomplished with a single rotation) or R and R' share a man (in which case the transformation from $\mathcal{M}_i$ to $\mathcal{M}_{i+1}$ does not move this man to his suitor in $\mathcal{M}_i$, so is not a rotation).

Once we find all of the rotations using Algorithm 2, we can order them to find the rotation poset P using the relation given in Definition 3.6.

Algorithm 2 Find-All-Rotations Algorithm

- Initially we start with the male-optimal stable matching $\mathcal{M}_0$, and some ordering of the men.
- In the current matching $\mathcal{M}_i$, among the ordered men, pick the first man who has a suitor. (If there are no men that have a suitor, then $\mathcal{M}_i$ is the female-optimal matching and the algorithm stops.) Let man M_1 be the first man who has a suitor, namely $S_{\mathcal{M}_i}(M_1)$.
- Start constructing the sequence $(M_1, w_1), (M_2, w_2), \ldots$, where $w_1 = sp_{\mathcal{M}_i}(M_1)$, for $l = 2, 3, \ldots, w_l = S_{\mathcal{M}_i}(M_{l-1})$ and $M_l = sp_{\mathcal{M}_i}(w_l)$.
- If there exists a $t \in \{1, \ldots, l-1\}$ such that $w_t = w_l$, then return the rotation $(M_t, w_t), \ldots, (M_l, w_l)$, and apply the rotation to $\mathcal{M}_i$ to get a new stable matching $\mathcal{M}_{i+1}$. Otherwise, increment l and continue constructing the sequence.

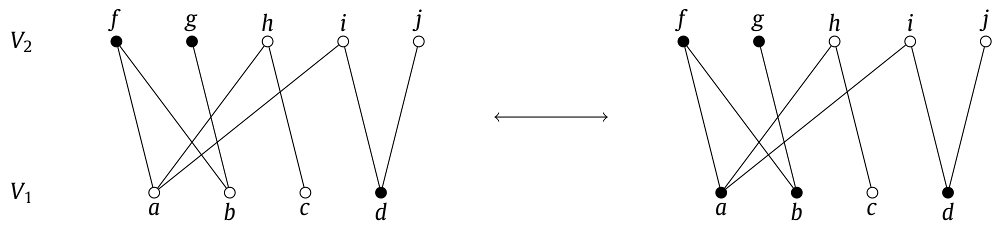


Fig. 1. The correspondence between independent sets and downsets.

3.5. #BIS, independent sets, and stable matchings

The rotation poset for a matching instance plays a key role in what follows. To prove Theorem 1, we take a #*BIS* instance $G = (V_1 \cup V_2, E)$ and view this as the rotation poset of a matching instance. In particular, G is the Hasse diagram of the poset when we draw G with the set V_2 "above" V_1.

Each independent set in the bipartite graph naturally corresponds to a downset in the partial order, and vice-versa. See Fig. 1 for an example. An independent set, namely $\{d, f, g\}$, is shown in the left of that figure. The corresponding downset is shown on the right. This downset is obtained by taking the set $\{d, f, g\}$ and adding the two elements a and b, as $a < f$ and $b < f$ (and $b < g$) in the Hasse diagram. Conversely, given a downset, such as the one on the right of the diagram, we can find the corresponding independent set in G by taking the set of maximal elements of the downset.

So given G, we then construct a matching instance (using 3-dimensional preference and attribute vectors) whose rotation poset is (isomorphic to) G, giving a 1–1 correspondence for our AP-reduction from #*BIS* to #*SM*(*k*-attribute), showing that #*BIS* $\leq_{AP}$ #*SM*(*k*-attribute). This construction, and the proof of the correspondence, is in Section 4.

The reverse implication #*SM*(*k*-attribute) $\leq_{AP}$ #*BIS* follows from the two results that #*SM* $\leq_{AP}$ #*Downsets* (Theorem 5, quoted here from [14]) and #*Downsets* $\leq$ #*BIS* [6, Lemma 9], where #*Downsets* is the problem of counting the number of downsets in a partial order.

4. Stable matchings in the *k*-attribute model ($k \geq 3$)

In this section we give our construction to show AP-reducibility from #*BIS* to the *k*-attribute stable matching model when $k \geq 3$.

Given our previous remarks about the relation between #*BIS*, independent sets, and stable matchings, our procedure is as follows:

1. Let $G = (V_1 \cup V_2, E)$ denote a bipartite graph where $|E| = n$. Our goal will be to construct a *k*-attribute stable matching instance for which we can show that the Hasse diagram of its rotation poset is G. This will give a bijection between stable matchings and downsets of G, hence a bijection between stable matchings and independent sets of G.
2. Using G, in the manner to be specified in Section 4.1, we construct preference lists for a 3-attribute stable matching instance with $3n$ men and $3n$ women.
3. Given this matching instance, we find the male-optimal and female-optimal matchings.
4. Using the **Find-All-Rotations** algorithm, we extract the rotations from our stable matching instance.
5. Having these rotations, we construct the partial order, P, on these rotations (specified by the transitive closure of the "explicitly precedes" relation).
6. We finally show that P is isomorphic to G (when G is viewed as a partial order), thereby showing our construction is an approximation-preserving reduction from #*BIS* to #*SM*(3-attribute).

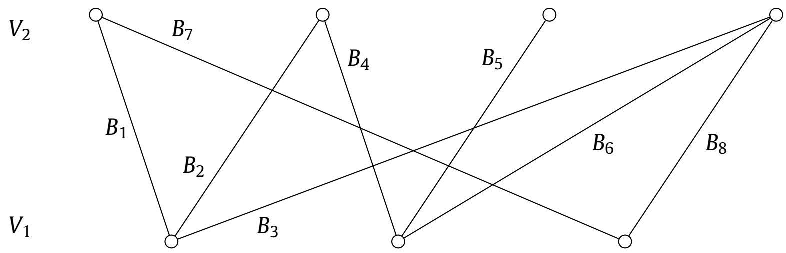


Fig. 2. A *BIS* instance and our labeling of its edges.

4.1. Construction of the stable matching instance

4.1.1. BIS and permutations

Let $G = (V_1 \cup V_2, E)$ denote our *BIS* instance, where $E \subseteq V_1 \times V_2$ and $|E| = n$.

Using G we will construct a 3-attribute stable matching instance with $3n$ men and $3n$ women. The men and women of the instance are denoted $\{A_1, \ldots, A_n, B_1, \ldots, B_n, C_1, \ldots, C_n\}$ and $\{a_1, \ldots, a_n, b_1, \ldots, b_n, c_1, \ldots, c_n\}$, respectively. To describe our construction, we label the edges of G B_1 through B_n from "left-to-right" with respect to the vertices (V_1) on the bottom. This becomes more clear from the example in Fig. 2. We refer to edge B_i as man B_i, and this will be clear from the context.

For our construction we associate two permutations, ρ and σ, of $[n] = \{1, \ldots, n\}$ with the BIS instance. The cycles of ρ correspond to vertices in V_1 and those of σ correspond to vertices in V_2. In other words, if the edges incident to a vertex in V_2 are $B_{i_1}, B_{i_2}, \ldots, B_{i_d}$, then $(i_1, i_2, \ldots, i_d)$ is a σ-cycle. We define ρ-cycles in a similar fashion. If G has $k = |V_1|$ vertices on the bottom and $l = |V_2|$ vertices on the top, then the permutations ρ and σ have k and l cycles, respectively. Since the graph G will turn out to be isomorphic to a rotation poset, every vertex in G will represent a rotation in the stable matching instance. The rotations of the stable matching instance will be governed by the ρ- and σ-cycles in a manner to be specified. The rotations corresponding to the ρ-cycles will be called ρ-rotations and those corresponding to the σ-cycles will be called σ-rotations.

In the example of Fig. 2, the three ρ-cycles are $\rho_1 = (1, 2, 3)$, $\rho_2 = (4, 5, 6)$, and $\rho_3 = (7, 8)$. The four σ-cycles are $\sigma_1 = (1, 7)$, $\sigma_2 = (2, 4)$, $\sigma_3 = (5)$, and $\sigma_4 = (3, 6, 8)$.

Here is a brief overview of how we go about constructing a stable matching instance from a given bipartite graph.

First of all, the male-optimal stable matching in our matching instance we construct will consist of the pairs (A_i, a_i), (B_i, b_i), (C_i, c_i) for all $i \in [n]$. (We must show later this is indeed the case for the construction we describe.)

A ρ-cycle of the form $(i_1, i_1 + 1, \ldots, i_2)$ will correspond to the ρ-rotation, R, of the form

$$\{(B_{i_1}, b_{i_1}), (A_{i_1}, a_{i_1}), (B_{i_1+1}, b_{i_1+1}), (A_{i_1+1}, a_{i_1+1}), \ldots, (B_{i_2}, b_{i_2}), (A_{i_2}, a_{i_2})\}.$$

This rotation R arises from a vertex $v \in V_1$ with edges $B_{i_1}, B_{i_1+1}, \ldots, B_{i_2}$ incident to it.

We will later show that a σ-rotation R' is of the form

$$\{(B_{i_1}, a_{i_1}), (C_{i_1}, c_{i_1}), (B_{i_2}, a_{i_2}), (C_{i_2}, c_{i_2}), \ldots, (B_{i_p}, a_{i_p}), (C_{i_p}, c_{i_p})\},$$

where $(i_1, i_2, \ldots, i_p)$ is the corresponding σ-cycle, and that the rotation R' corresponds to the vertex $v' \in V_2$ with edges $B_{i_1}, B_{i_2}, \ldots, B_{i_p}$ incident to it.

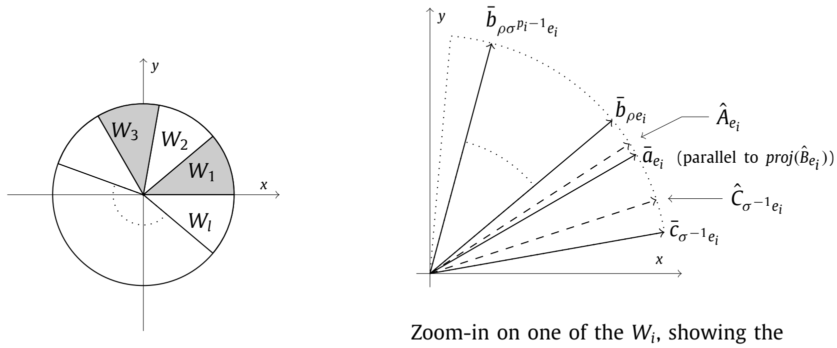
In this manner, every rotation in the rotation poset is defined in terms of the men involved in them, the women being the (then-current) partners of the men that are in the rotation. Assuming that the above two claims regarding rotations are valid (as we will show below), we make the following observation.

Observation 7. *A ρ-cycle and a σ-cycle can have at most one element in common. (This is because G is a graph and not a multi-graph.) This means that a ρ-rotation and a σ-rotation can have at most one man in common. This similarly holds for the women.*

In the next section we start by assigning preference vectors and position vectors to the men and women in our stable matching instance. Following that, we construct the initial portion of their preference lists. We then find the male- and the female-optimal matchings using the Gale–Shapley algorithm. After finding the male- and the female-optimal matchings, we extract all the rotations of the rotation poset using the **Find-All-Rotations** algorithm. Finally, we obtain the rotation poset by ordering rotations using the *explicitly precedes* relation. As we stated earlier, we will find this rotation poset is isomorphic to G, showing our construction is a mapping from the set of #*BIS* instances to #*SM*(3-attribute) instances.

4.1.2. Assigning preference and position vectors

Suppose $D_1, \ldots, D_l$ are the l cycles of σ of lengths $p_1, \ldots, p_l$, respectively. Let e_i be a representative element of cycle D_i. In other words, we can represent the σ-cycle D_i as $D_i = \big(e_i, \sigma(e_i), \ldots, \sigma^{p_i-1}(e_i)\big)$. (We may, for example, select e_i to be the smallest number in the cycle, and we will do so here). In what follows we will often abbreviate $\sigma x = \sigma(x)$, $\sigma^2 x = \sigma^2(x)$, $\sigma^{-1}x = \sigma^{-1}(x)$, etc, and, similarly, $\rho x = \rho(x)$, etc.



Arrange (projections of) position vectors for each W_i inside the sectors shown

Zoom-in on one of the W_i, showing the women's position vectors and men's preference vectors

Fig. 3. Placement of the women's position vectors and men's preference vectors.

Let $Rep(\sigma) = \{e_1, e_2 \ldots, e_l\}$ be the set of representative elements we choose for the cycles of σ. Let $W_i = \{a_x : x \in D_i\} \cup \{b_{\rho x} : x \in D_i\} \cup \{c_{\sigma^{-1}x} : x \in D_i\}$. Let $T(x) = \{c_{\sigma^{-1}x}, a_x, b_{\rho x}\}$, where $x \in D_i$. It follows that $W_i = \cup_{x \in D_i} T(x)$ and $T(i) \cap T(j) = \emptyset$ for $i \neq j$.

Using the definitions above, we begin to create a stable matching instance in the 3-attribute model whose rotation poset is the graph G. As a reminder, every man, say A_i, is associated with two vectors: (i) a position vector denoted by $\bar{A}_i$, and (ii) a preference vector denoted by $\hat{A}_i$. Every woman similarly has her own position and preference vectors. Each man ranks the women based on the dot product of his preference vector with their position vectors. In other words, if $\hat{A}_i \cdot \bar{b} > \hat{A}_i \cdot \bar{c}$, then man A_i prefers woman b over c. Note that we can always assign preference vectors so that $|\hat{A}_i| = 1$ (by normalizing those vectors).

Our task, therefore, is to specify the position and preference vectors for all the men and women in our matching instance.

First we fix the position vectors of the women. The z-coordinate of women a_i and c_i is set to 0 for $1 \leq i \leq n$. The z-coordinate of woman b_i is set to 4^i for $1 \leq i \leq n$. The x- and y-coordinates of a_i, b_i, and c_i are such that the projection of each women's position vector onto the x-y plane lies on the unit circle $x^2 + y^2 = 1$. Furthermore, we group the projections according to the sets W_i. In other words, all women in W_i are embedded in an angle of ϵ on the unit circle, where $\epsilon = 2\pi/n^2$. These groups are embedded around the circle in the order W_1 through W_l, and the angle between two adjacent groups is $(2\pi - l\epsilon)/l$. Note that W_l is adjacent to W_{l-1} and W_1. Group W_i starts at angle $2\pi(i-1)/l$ and ends at $2\pi(i-1)/l + \epsilon$.

Within the group W_i, the women are further sub-grouped into triplets

$$T(e_i), T(\sigma(e_i)), \ldots, T(\sigma^{p_i-1}(e_i)).$$

Within the angle of size ϵ, the sub-groups are embedded in the order $T(e_i)$ through $T(\sigma^{p_i-1}(e_i))$, with each $T(\cdot)$ spanning an angle of $6\theta_i$. The angle between two adjacent $T(\cdot)$'s is θ_i, where $\theta_i = \epsilon/(7p_i - 1)$. Within each $T(x)$, the women appear in the order $c_{\sigma^{-1}x}$, a_x, and $b_{\rho x}$, and the angle between $\bar{c}_{\sigma^{-1}x}$ and $\bar{a}_x$ is $4\theta_i$, and the angle between $\bar{a}_x$ and $\bar{b}_{\rho x}$ is $2\theta_i$ (see Fig. 3). We summarize the above description by giving the exact coordinates for the position vector for the women.

$$\begin{aligned}
&\text{Let } \epsilon = \frac{2\pi}{n^2}.\\
&\text{For } e_i \in Rep(\sigma),\ \text{ let } \theta_i = \epsilon/(7p_i - 1).\ \text{ Then for } 0 \leq m \leq p_i - 1 \text{ define}\\
&\bar{a}_{\sigma^m e_i} = (\cos(2\pi(i-1)/l + 7m\theta_i + 4\theta_i),\ \ \sin(2\pi(i-1)/l + 7m\theta_i + 4\theta_i),\ \ 0)\,,\\
&\bar{b}_{\rho\sigma^m e_i} = (\cos(2\pi(i-1)/l + 7m\theta_i + 6\theta_i),\ \ \sin(2\pi(i-1)/l + 7m\theta_i + 6\theta_i),\ \ 4^{\rho\sigma^m e_i}),\ \text{ and}\\
&\bar{c}_{\sigma^{m-1}e_i} = (\cos(2\pi(i-1)/l + 7m\theta_i),\ \ \sin(2\pi(i-1)/l + 7m\theta_i),\ \ 0)\,.
\end{aligned}$$

Next we define the preference vectors of the men. The z-coordinates of all $\hat{A}_i$ and $\hat{C}_i$ are set to 0. We place $\hat{A}_i$ between $\bar{a}_i$ and the projection onto the x-y plane of $\bar{b}_{\rho i}$. If the angle between $\bar{a}_i$ and (the projection of) $\bar{b}_{\rho i}$ is α, then the angle between $\bar{a}_i$ and $\hat{A}_i$ is $\frac{1}{3}\alpha$, and the angle between $\hat{A}_i$ and (the projection of) $\bar{b}_{\rho i}$ is $\frac{2}{3}\alpha$. This will ensure that A_i prefers a_i over $b_{\rho i}$. We will later show that the preference list of A_i starts with $a_i b_{\rho i}$. We place $\hat{C}_i$ between $\bar{c}_i$ and $\bar{a}_{\sigma i}$ such that if the angle between $\bar{c}_i$ and $\bar{a}_{\sigma i}$ is β, then the angle between $\bar{c}_i$ and $\hat{C}_i$ is $\frac{2}{5}\beta$ and the angle between $\hat{C}_i$ and $\bar{a}_{\sigma i}$ is $\frac{3}{5}\beta$. This will ensure that C_i prefers c_i over $a_{\sigma i}$. We will later show that the preference list of C_i starts with $c_i a_{\sigma i}$.

Finally, we place $\hat{B}_i$, which is of unit length, such that $\hat{B}_i$ makes an angle of $\phi = 2\pi/100$ with the vertical axis (z-axis) and its projection on the x-y plane is parallel to $\bar{a}_i$. In other words, the projection of $\hat{B}_i$ on the $z = 0$ plane is $\sin\phi\ \bar{a}_i$. We summarize the above discussion by providing the exact coordinates of $\hat{A}_i$, $\hat{B}_i$, and $\hat{C}_i$.

Let $\phi = 2\pi/100$ and $\epsilon = \dfrac{2\pi}{n^2}$.

For $e_i \in Rep(\sigma)$, let $\theta_i = \epsilon/(7p_i - 1)$. Then for $0 \le m \le p_i - 1$ define

$$\hat{A}_{\sigma^m e_i} = (\cos(2\pi(i-1)/l + 7m\theta_i + (14/3)\theta_i),\quad \sin(2\pi(i-1)/l + 7m\theta_i + (14/3)\theta_i),\quad 0),$$

$$\hat{B}_{\sigma^m e_i} = (\sin\phi\cos(2\pi(i-1)/l + 7m\theta_i + 4\theta_i),\quad \sin\phi\sin(2\pi(i-1)/l + 7m\theta_i + 4\theta_i),\quad \cos\phi),$$

and

$$\hat{C}_{\sigma^{m-1} e_i} = (\cos(2\pi(i-1)/l + 7m\theta_i + (8/5)\theta_i),\quad \sin(2\pi(i-1)/l + 7m\theta_i + (8/5)\theta_i),\quad 0).$$

In Section 4.1.3 we establish the preference lists of the men and women. The vectors given above let us determine the preference lists of the men, so we now specify the position vectors of the men and the preference vectors of the women. This proceeds in a similar manner as above.

Suppose E_1 through E_k are the k cycles of ρ of lengths q_1 through q_k, respectively. As above, let f_i be a representative element of cycle E_i, so that we can write the ρ-cycle as $\big(f_i, \rho(f_i), \ldots, \rho^{q_i-1}(f_i)\big)$. Let $Rep(\rho) = \{f_1, f_2 \ldots, f_k\}$ be the set of representative elements we select for the cycles of ρ. Let $U_i = \{A_{\rho^{-1}r} : r \in E_i\} \cup \{B_r : r \in E_i\} \cup \{C_r : r \in E_i\}$. Let $S(r) = \{A_{\rho^{-1}r}, B_r, C_r\}$, where $r \in E_i$. It follows that $U_i = \cup_{r\in E_i} S(r)$ and $S(i) \cap S(j) = \emptyset$ for $i \ne j$.

We fix the position vectors of the men. The placement of the men is similar to that of the women. The z-coordinate of the men A_i and B_i is set to 0 for $1 \le i \le n$. The z-coordinate of man C_i is set to 4^i for $1 \le i \le n$. The x- and y-coordinates of A_i, B_i and C_i are such that the projection of the men onto the $z = 0$ plane lies on the unit circle $x^2 + y^2 = 1$. Similar to above, the projections are grouped according to the sets U_i. In other words, with $\epsilon = 2\pi/n^2$, all men in U_i are embedded in an angle of ϵ on the unit circle. The groups are embedded around the circle in the order U_1 through U_k and the angle between two adjacent groups is $(2\pi - k\epsilon)/k$. Note that U_k is adjacent to U_{k-1} and U_1. The group U_i starts at angle $2\pi(i-1)/k$ and ends at $2\pi(i-1)/k + \epsilon$. Within the group U_i, the men are further sub-grouped into triplets $S(f_i), S(\rho(f_i)), \ldots, S(\rho^{q_i-1}(f_i))$. Within the angle of ϵ, the sub-groups are embedded in the order $S(f_i)$ through $S(\rho^{q_i-1}(f_i))$, with each $S(\cdot)$ spanning an angle of $6\omega_i$, where the angle between two adjacent $S(\cdot)$'s is $\omega_i = \epsilon/(7q_i - 1)$. Within each $S(j)$, the men appear in the order $A_{\rho^{-1}j}, B_j$ and C_j, and the angle between $\bar{A}_{\rho^{-1}j}$ and $\bar{B}_j$ is $4\omega_i$ and the angle between $\bar{B}_j$ and $\bar{C}_j$ is $2\omega_i$. Here are the exact coordinates for the position vector of each man.

Let $\epsilon = \dfrac{2\pi}{n^2}$.

For $f_i \in Rep(\rho)$, let $\omega_i = \epsilon/(7q_i - 1)$. Then for $0 \le m \le q_i - 1$ we define

$$\bar{A}_{\rho^{m-1} f_i} = (\cos(2\pi(i-1)/k + 7m\omega_i),\quad \sin(2\pi(i-1)/k + 7m\omega_i),\quad 0),$$

$$\bar{B}_{\rho^m f_i} = (\cos(2\pi(i-1)/k + 7m\omega_i + 4\omega_i),\quad \sin(2\pi(i-1)/k + 7m\omega_i + 4\omega_i),\quad 0), \quad \text{and}$$

$$\bar{C}_{\rho^m f_i} = (\cos(2\pi(i-1)/k + 7m\omega_i + 6\omega_i),\quad \sin(2\pi(i-1)/k + 7m\omega_i + 6\omega_i),\quad 4^{\rho^m f_i}).$$

Finally, we define the preference vectors of the women. The z-coordinates of $\hat{b}_i$ and $\hat{c}_i$ are set to 0. Suppose the angle between $\bar{B}_i$ and (the projection onto the x-y plane of) $\bar{C}_i$ is α. Then we place $\hat{c}_i$ in the x–y plane between $\bar{B}_i$ and (the projection of) $\bar{C}_i$ such that the angle between $\bar{B}_i$ and $\hat{c}_i$ is $\frac{1}{3}\alpha$, and the angle between $\hat{c}_i$ and (the projection of) $\bar{C}_i$ is $\frac{2}{3}\alpha$. We place $\hat{b}_i$ between $\bar{A}_{\rho^{-1}i}$ and $\bar{B}_i$ such that if the angle between $\bar{A}_{\rho^{-1}i}$ and $\bar{B}_i$ is β, then the angle between $\bar{A}_{\rho^{-1}i}$ and $\hat{b}_i$ is $\frac{2}{5}\beta$ and the angle between $\hat{b}_i$ and $\bar{B}_i$ is $\frac{3}{5}\beta$.

We place $\hat{a}_i$, which is of unit length, such that $\hat{a}_i$ makes an angle of $\phi = 2\pi/100$ with the vertical axis (z-axis) and its projection on the $z = 0$ plane is parallel to $\bar{B}_i$. In other words, the projection of $\hat{a}_i$ on the $z = 0$ plane is $\sin\phi\ \bar{B}_i$. Therefore, the exact coordinates of the preference vectors $\hat{c}_i$, $\hat{a}_i$ and $\hat{b}_i$ are as follows.

Let $\phi = 2\pi/100$ and $\epsilon = \dfrac{2\pi}{n^2}$.

For $f_i \in Rep(\rho)$, let $\omega_i = \epsilon/(7q_i - 1)$. Then for $0 \le m \le q_i - 1$ we define

$$\hat{a}_{\rho^m f_i} = (\sin\phi\cos(2\pi(i-1)/k + 7m\omega_i + 4\omega_i),\ \sin\phi\sin(2\pi(i-1)/k + 7m\omega_i + 4\omega_i),\ \cos\phi),$$

$$\hat{b}_{\rho^m f_i} = (\cos(2\pi(i-1)/k + 7m\omega_i + (8/5)\omega_i),\quad \sin(2\pi(i-1)/k + 7m\omega_i + 8/5\omega_i),\quad 0), \quad \text{and}$$

$$\hat{c}_{\rho^m f_i} = (\cos(2\pi(i-1)/k + 7m\omega_i + 14/3\omega_i),\quad \sin(2\pi(i-1)/k + 7m\omega_i + (14/3)\omega_i),\quad 0).$$

4.1.3. *Constructing (partial) preference lists*

Using the vectors defined in the previous section, we now examine the preference lists of the men and women of our constructed instance.

First we will establish that the preference lists of A_i and C_i start with $a_i b_{\rho i}$ and $c_i a_{\sigma i}$, respectively. Since $\hat{A}_i$ and $\hat{C}_i$ have a z-component that is equal to zero, it is enough to consider the projections of $\bar{a}_i$, $\bar{b}_i$ and $\bar{c}_i$ on the x-y plane. Furthermore, since

$\hat{A}_i$, $\hat{C}_i$, and the projections of $\bar{a}_i$, $\bar{b}_i$ and $\bar{c}_i$ are all of unit length, the dot product is essentially a function of the angle between the two vectors. In other words, if the angle between $\hat{A}_i$ and $\bar{b}$ is greater than the angle between $\hat{A}_i$ and $\bar{c}$, then $\hat{A}_i \cdot \bar{b} < \hat{A}_i \cdot \bar{c}$. Recalling that $\hat{A}_i$ lies between $\bar{a}_i$ and $\bar{b}_{\rho i}$, and is closer to $\bar{a}_i$, then a_i will appear first on the preference list of A_i. The women positioned next to a_i on the unit circle are $b_{\rho i}$ and $c_{\sigma^{-1}i}$. Since the angle between $\bar{a}_i$ and $\bar{c}_{\sigma^{-1}i}$ is twice the angle between $\bar{a}_i$ and $\bar{b}_{\rho i}$ (and $\hat{A}_i$ lies between $\bar{a}_i$ and $\bar{b}_{\rho i}$), we see that woman $b_{\rho i}$ appears second on the preference list of A_i.

Since $\hat{C}_i$ lies between $\bar{c}_i$ and $\bar{a}_{\sigma i}$ and is closer to $\bar{c}_i$, we see that c_i will appear first on the preference list of C_i. By construction, the angle between $\bar{c}_i$ and $\bar{a}_{\sigma i}$ is $4\theta_i$. We note that the angle between $\hat{C}_i$ and $\bar{c}_i$ is $2/5(4\theta_i) = 8/5\ \theta_i$ and that between $\hat{C}_i$ and $\bar{a}_{\sigma i}$ is $3/5(4\theta_i) = 12/5\ \theta_i$. We consider two cases (i) c_i is not the first woman in W_j for $1 \le j \le l$, i.e. $\sigma i \notin Rep(\sigma)$, (ii) c_i is the first woman in some W_j for $1 \le j \le l$, i.e. $\sigma i = e_j$ for some $1 \le j \le l$.

Case(i) As c_i is not the first woman in W_j for $1 \le j \le l$, the women positioned next to c_i are $a_{\sigma i}$ and $b_{\rho i}$ where $c_i \in T(\sigma i)$ and $b_{\rho i} \in T(i)$. The angle between (the projection of) $\bar{b}_{\rho i}$ and $\bar{c}_i$ is the angle between two adjacent $T(\cdot)$'s, which is one-fourth of the angle between $\bar{c}_i$ and $\bar{a}_{\sigma i}$, i.e. $1/4(4\theta_i) = \theta_i$. Hence, the angle between $\hat{C}_i$ and $b_{\rho i}$ is $\theta_i + 8/5\ \theta_i = 13/5\ \theta_i > 12/5\ \theta_i$. Hence, $a_{\sigma i}$ appears second on the preference list of C_i.

Case(ii) As c_i is the first woman in some W_j for $1 \le j \le l$, the women positioned next to c_i are $a_{\sigma i}$ and b_x, where $c_i, a_{\sigma i} \in W_j$ and $b_x \in W_{j-1}$. Note that $j - 1 \stackrel{\text{def}}{=} l$ if $j = 1$. The angle between $\bar{c}_i$ and $\bar{a}_{\sigma i}$ is at most ϵ and the angle between $\bar{c}_i$ and $\bar{b}_x$ is the angle between W_{j-1} and W_j, which is $(2\pi - l\epsilon)/l = 2\pi/l - \epsilon > \epsilon$. Hence, $a_{\sigma i}$ appears second on the preference list of C_i.

Lastly we examine the preference list of B_i. We will show that the relative order of the b-women on the preference list of B_i is $b_n b_{n-1} b_{n-2} \cdots b_1$ for all $1 \le i \le n$ and that B_i prefers b_1 over any woman $w \notin \{b_1, \ldots, b_n\}$. This will imply that the preference list of B_i starts with $b_n b_{n-1} \cdots b_1$ for all $1 \le i \le n$. The dot product of $\hat{B}_i$ and $\bar{b}_j$ is

$$\hat{B}_i \cdot \bar{b}_j = \sin\phi\ \bar{a}_i \cdot \bar{b}_j + \cos\phi\ 4^j.$$

$$\text{Hence, } \cos\phi\ 4^j - \sin\phi \le \hat{B}_i \cdot \bar{b}_j \le \cos\phi\ 4^j + \sin\phi, \quad \text{and}$$

$$\cos\phi\ 4^j - \phi \le \hat{B}_i \cdot \bar{b}_j \le \cos\phi\ 4^j + \phi.$$

Comparing $\hat{B}_i \cdot \bar{b}_j$ with $\hat{B}_i \cdot \bar{b}_{j+1}$, we observe that

$$\hat{B}_i \cdot \bar{b}_j \le \cos\phi\ 4^j + \phi < \cos\phi\ 4^{j+1} - \phi \le \hat{B}_i \cdot \bar{b}_{j+1} \quad (\text{since } \phi = 2\pi/100,\ \cos\phi > 3/4).$$

This implies that the relative order of the b-women on the preference list of B_i is $b_n b_{n-1} b_{n-2} \cdots b_1$ for all $1 \le i \le n$.

Next we show that B_i prefers b_1 over any woman $w \notin \{b_1, \ldots, b_n\}$. Every woman $w \notin \{b_1, \ldots, b_n\}$ lies in the x-y plane. Hence, it is enough to consider the projection of $\hat{B}_i$ in the plane, which is $\sin\phi\ \bar{a}_i$. Comparing $\hat{B}_i \cdot \bar{b}_1$ with $\hat{B}_i \cdot \bar{a}_x$ and $\hat{B}_i \cdot \bar{c}_x$ for $1 \le x \le n$, we observe that

$$\hat{B}_i \cdot \bar{b}_1 \ge \cos\phi\ 4 - \phi > (3/4) \cdot 4 - \phi > 2,$$

$$\hat{B}_i \cdot \bar{a}_x = \sin\phi\ \bar{a}_i \cdot \bar{a}_x \le \sin\phi \le \phi < 1 < \hat{B}_i \cdot \bar{b}_1, \quad \text{and}$$

$$\hat{B}_i \cdot \bar{c}_x = \sin\phi\ \bar{a}_i \cdot \bar{c}_x \le \sin\phi \le \phi < 1 < \hat{B}_i \cdot \bar{b}_1.$$

Hence, we have that the preference list of B_i starts with $b_n b_{n-1} b_{n-2} \cdots b_1$ for $1 \le i \le n$. Next we show that B_i prefers a_i over any woman $w \notin \{a_i, b_1, \ldots, b_n\}$. Comparing $\hat{B}_i \cdot \bar{a}_i$ with $\hat{B}_i \cdot \bar{a}_x$, where $x \ne i$ and $\hat{B}_i \cdot \bar{c}_j$ for $1 \le j \le n$, we find that

$$\hat{B}_i \cdot \bar{a}_i = \sin\phi\ \bar{a}_i \cdot \bar{a}_i = \sin\phi,$$

$$\hat{B}_i \cdot \bar{a}_x = \sin\phi\ \bar{a}_i \cdot \bar{a}_x < \sin\phi = \hat{B}_i \cdot \bar{a}_i \quad (\text{since } \bar{a}_i \cdot \bar{a}_x < 1 \text{ for } i \ne x), \quad \text{and}$$

$$\hat{B}_i \cdot \bar{c}_x = \sin\phi\ \bar{a}_i \cdot \bar{c}_x < \sin\phi = \hat{B}_i \cdot \bar{a}_i.$$

Now the preference list of B_i reads $b_n b_{n-1} b_{n-2} \cdots b_1 a_i$ for $1 \le i \le n$. Finally, we consider two cases — (i) a_i is not the last a-woman in any W_j where $1 \le j \le l$ (ii) a_i is the last a-woman in some W_j where $1 \le j \le l$.

Case (i) Suppose $a_i \in W_j$ for some $j \in \{1, \ldots, l\}$. As a_i is not the last a-woman in W_j, the next a-woman in W_j is $a_{\sigma i}$. In other words, $a_i \in T(i)$ and $a_{\sigma i} \in T(\sigma i)$, where $T(i) = \{c_{\sigma^{-1}i}, a_i, b_{\rho i}\}$ and $T(\sigma i) = \{c_i, a_{\sigma i}, b_{\rho\sigma i}\}$. The angle between $\bar{c}_{\sigma^{-1}i}$ and $\bar{a}_i$ is $4\theta_j$, and that between $\bar{a}_i$ and (the projection of) $\bar{b}_{\rho i}$ is $2\theta_j$. The angle between $\bar{b}_{\rho i}$ and $\bar{c}_i$ is the angle between $T(i)$ and $T(\sigma i)$, which is θ_j. Hence, the angle between $\bar{a}_i$ and $\bar{c}_i$ is $2\theta_j + \theta_j = 3\theta_j$. Note that the projection of the b-women onto the unit circle is irrelevant as they have already been ranked by B_i. Hence, we need only consider the a-women and the c-women. Given the placement of the preference vector $\hat{B}_i$, after a_i, B_i will prefer either $c_{\sigma^{-1}i}$ or c_i. Comparing the dot product of $\hat{B}_i$ with $\bar{c}_{\sigma^{-1}i}$ and with $\bar{c}_i$, we get

$$\hat{B}_i \cdot \bar{c}_i = \sin\phi\ \bar{a}_i \cdot \bar{c}_i = \sin\phi\cos 3\theta_j, \quad \text{and}$$

$$\hat{B}_i \cdot \bar{c}_{\sigma^{-1}i} = \sin\phi\ \bar{a}_i \cdot \bar{c}_{\sigma^{-1}i} = \sin\phi\cos 4\theta_j < \sin\phi\cos 3\theta_j = \hat{B}_i \cdot \bar{c}_i.$$

Hence, the preference list of B_i reads $b_n b_{n-1} b_{n-2} \cdots b_1 a_i c_i$.

Case (ii) Suppose $a_i \in W_j$ for some $j \in \{1, \ldots, l\}$ *and* a_i is the last a-woman in the group. This implies that $a_i \in T(i) = \{c_{\sigma^{-1}i}, a_i, b_{\rho i}\}$ and $T(i)$ is the last sub-group of W_j. Since $T(i)$ is the last sub-group of W_j, W_j starts with the sub-group $T(\sigma i) = \{c_i, a_{\sigma i}, b_{\rho\sigma i}\}$ followed by $T(\sigma^2 i), \ldots, T(\sigma^{p_j-1}i), T(\sigma^{p_j}i) = T(i)$. The angle subtended by the group W_j at the origin is ϵ and $(2\pi - l\epsilon)/l$ is the angle between two adjacent W groups. Hence, comparing the dot product of $\hat{B}_i$ with any a-woman or c-woman in W_j with any a-woman or c-woman from W_x where $x \neq j$, we obtain

$$\begin{aligned}
&w_1 \in W_j,\ w_2 \in W_x, x \neq j, \quad w_1, w_2 \notin \{b_1, \ldots, b_n\}\\
&\hat{B}_i \cdot \bar{w}_1 = \sin\phi\ \bar{a}_i \cdot \bar{w}_1 \geq \sin\phi\ \cos\epsilon, \quad \text{and}\\
&\hat{B}_i \cdot \bar{w}_2 = \sin\phi\ \bar{a}_i \cdot \bar{w}_2 \leq \sin\phi\ \cos((2\pi - l\epsilon)/l) < \sin\phi\ \cos\epsilon \leq \hat{B}_i \cdot \bar{w}_1.
\end{aligned}$$

We conclude that B_i prefers the a-women and c-women in W_j over any a-woman or c-woman in any other group. Within W_j, the $T(\cdot)$ sub-groups occur in the order $T(i), T(\sigma i), \ldots, T(\sigma^{p_j-1}i)$ and the projection of $\hat{B}_i$ lies inside $T(i)$. We remind the reader that within $T(\sigma^m i)$, where $1 \leq m \leq p_j$, the angle between $\bar{c}_{\sigma^{m-1}i}$ and $\bar{a}_{\sigma^m i}$ is $4\theta_j$, the angle between $\bar{a}_{\sigma^m i}$ and $\bar{b}_{\rho\sigma^m i}$ is $2\theta_j$, and that between two adjacent $T(\cdot)$'s is θ_j, where $(7p_j - 1)\theta_j = \epsilon$. This implies that for $1 \leq m \leq p_j$ the angle between $\bar{a}_{\sigma^m i}$ and the projection of $\hat{B}_i$ (which is the angle between $\bar{a}_{\sigma^m i}$ and $\bar{a}_i$) is $(p_j - m)7\theta_j$. For $1 \leq m \leq p_j$, the angle between $\bar{c}_{\sigma^{m-1}i}$ and the projection of $\hat{B}_i$ (which is the angle between $\bar{c}_{\sigma^{m-1}i}$ and $\bar{a}_i$) is $(p_j - m)7\theta_j + 4\theta_j = (7(p_j - m) + 4)\theta_j$. Now computing the dot product of $\hat{B}_i$ with $\bar{a}_{\sigma^m i}$ and $\bar{c}_{\sigma^{m-1}i}$, we see that

$$\begin{aligned}
&\text{for } 1 \leq m \leq p_j, \quad \theta_j = \epsilon/(7p_j - 1), \text{ we have}\\
&\hat{B}_i \cdot \bar{a}_{\sigma^m i} = \sin\phi\ \bar{a}_i \cdot \bar{a}_{\sigma^m i} = \sin\phi\ \cos(7(p_j - m)\theta_j),\\
&\hat{B}_i \cdot \bar{c}_{\sigma^{m-1}i} = \sin\phi\ \bar{a}_i \cdot \bar{c}_{\sigma^{m-1}i} = \sin\phi\ \cos((7(p_j - m) + 4)\theta_j),\\
&\hat{B}_i \cdot \bar{c}_{\sigma^{m-1}i} = \sin\phi\ \cos((7(p_j - m) + 4)\theta_j) < \sin\phi\ \cos(7(p_j - m)\theta_j)\\
&\qquad = \hat{B}_i \cdot \bar{a}_{\sigma^m i}, \quad 1 \leq m \leq p_j, \quad \text{and}\\
&\hat{B}_i \cdot \bar{a}_{\sigma^m i} = \sin\phi\ \cos(7(p_j - m)\theta_j) < \sin\phi\ \cos((7(p_j - m - 1) + 4)\theta_j)\\
&\qquad = \hat{B}_i \cdot \bar{c}_{\sigma^m i}, \quad 1 \leq m \leq p_j - 1.
\end{aligned}$$

Combining the above inequalities and using the fact that $i = \sigma^{-1}e_j = \sigma^{p_j-1}e_j$, we get

$$\hat{B}_i \cdot \bar{c}_i < \hat{B}_i \cdot \bar{a}_{\sigma i} < \hat{B}_i \cdot \bar{c}_{\sigma i} < \cdots < \hat{B}_i \cdot \bar{c}_{\sigma^{(p_j-1)}i} < \hat{B}_i \cdot \bar{a}_{\sigma^{p_j}i} = \hat{B}_i \cdot \bar{a}_i.$$

Hence, the preference list of B_i is $b_n b_{n-1} b_{n-2} \cdots b_1 a_i c_{\sigma^{(p_j-1)}i} a_{\sigma^{(p_j-1)}i} \cdots a_{\sigma^2 i} c_{\sigma i} a_{\sigma i} c_i$.

We remind the reader that $e_i \in Rep(\sigma)$ is a representative element of cycle D_i and W_i is partitioned into $T(e_i), T(\sigma e_i), \ldots, T(\sigma^{(p_i-1)}e_i)$, where the sub-groups are embedded on the unit circle in the order $T(e_i)$ through $T(\sigma^{(p_i-1)}e_i)$, with $T(\sigma^{(p_i-1)}e_i)$ being the last sub-group in W_i. We now have the initial part of the preference lists of A_i, C_i and B_i. They are as follows:

$$\begin{aligned}
&\text{for } e_i \in Rep(\sigma), \\
A_{\sigma^m e_i} &: \quad a_{\sigma^m e_i} b_{\rho\sigma^m e_i}, \quad 0 \leq m \leq p_i - 1,\\
C_{\sigma^{(m-1)} e_i} &: \quad c_{\sigma^{(m-1)} e_i} a_{\sigma^m e_i}, \quad 0 \leq m \leq p_i - 1,\\
B_{\sigma^m e_i} &: \quad b_n b_{n-1} b_{n-2} \cdots b_1 a_{\sigma^m e_i} c_{\sigma^m e_i}, \quad 0 \leq m \leq p_i - 2, \quad \text{and}\\
B_{\sigma^{(p_i-1)} e_i} &: \quad b_n b_{n-1} b_{n-2} \cdots b_1 a_{\sigma^{(p_i-1)} e_i} c_{\sigma^{(p_i-2)} e_i} a_{\sigma^{(p_i-2)} e_i} \cdots a_{\sigma e_i} c_{e_i} a_{e_i} c_{\sigma^{(p_i-1)} e_i}.
\end{aligned} \tag{2}$$

The partial preference lists of the women can be obtained by arguments similar to those used for obtaining the men's preference lists. The partial preference lists for the women are as follows:

$$\begin{aligned}
&\text{for } f_i \in Rep(\rho), \\
b_{\rho^m f_i} &: \quad A_{\rho^{(m-1)} f_i} B_{\rho^m f_i}, \quad 0 \leq m \leq q_i - 1,\\
c_{\rho^m f_i} &: \quad B_{\rho^m f_i} C_{\rho^m f_i}, \quad 0 \leq m \leq q_i - 1,\\
a_{\rho^m f_i} &: \quad C_n C_{n-1} \cdots C_1 B_{\rho^m f_i} A_{\rho^m f_i}, \quad 0 \leq m \leq q_i - 2, \quad \text{and}\\
a_{\rho^{(q_i-1)} f_i} &: \quad C_n C_{n-1} \cdots C_1\\
&\qquad B_{\rho^{(q_i-1)} f_i} A_{\rho^{(q_i-2)} f_i} B_{\rho^{(q_i-2)} f_i} \cdots B_{\rho^2 f_i} A_{\rho f_i} B_{\rho f_i} A_{f_i} B_{f_i} A_{\rho^{(q_i-1)} f_i}.
\end{aligned} \tag{3}$$

We note that we have not specified the entire preference lists for the men and women. The remaining portion of each preference list appears *after* the part that we have given above, and there will never be any stable pairs involving a man/woman pair that is not shown on the partial preference lists given. The partial lists we have given are sufficient to find the male- and female-optimal matchings, and they contain the necessary information to generate *all* of the stable matchings for our constructed instance, or equivalently, to find all of the rotations for this instance.

4.2. Male- and female-optimal matchings

The rest of our analysis will use the partial preference lists in (2) and (3) and will not otherwise depend upon the position and preference vectors. In order to re-use our analysis in Section 6, we will be less specific about the men's partial preference lists (2). Let τ be a permutation of $\{1, \ldots, n\}$. Note that the men's partial preference lists from (2) correspond to the following partial preference lists by taking the permutation τ to be the identity permutation.

$$\begin{aligned}
&\text{For } e_i \in Rep(\sigma), \\
A_{\sigma^m e_i} \quad &: \quad a_{\sigma^m e_i} b_{\rho\sigma^m e_i}, \quad 0 \le m \le p_i - 1, \\
C_{\sigma^{(m-1)} e_i} \quad &: \quad c_{\sigma^{(m-1)} e_i} a_{\sigma^m e_i}, \quad 0 \le m \le p_i - 1, \\
B_{\sigma^m e_i} \quad &: \quad b_{\tau(n)} b_{\tau(n-1)} \cdots b_{\tau(1)} a_{\sigma^m e_i} c_{\sigma^m e_i}, \quad 0 \le m \le p_i - 2, \quad \text{and} \\
B_{\sigma^{(p_i-1)} e_i} \quad &: \quad b_{\tau(n)} b_{\tau(n-1)} \cdots b_{\tau(1)} a_{\sigma^{(p_i-1)} e_i} c_{\sigma^{(p_i-2)} e_i} a_{\sigma^{(p_i-2)} e_i} \cdots a_{\sigma e_i} c_{e_i} a_{e_i} c_{\sigma^{(p_i-1)} e_i}.
\end{aligned} \tag{4}$$

The rest of our analysis will use the partial preference lists (3) and (4), We will not make any assumptions about the permutation τ even though, for the purposes of this section, we could assume that it is the identity permutation.

We will find the male-optimal and female-optimal stable matchings using the Gale–Shapley algorithm. Recall that the order in which the men propose does not matter and any order always leads to the male-optimal matching (provided a man proposes to the highest-ranked woman (on his preference list) who hasn't yet rejected him). Therefore, we may suppose the men propose in the order $\{A_1, \ldots, A_n, C_1, \ldots, C_n, B_{\tau(n)}, \ldots, B_{\tau(1)}\}$.

For $1 \le i \le n$, men A_i and C_i are paired up with their first choices, women a_i and c_i respectively, as each of these women will receive exactly one proposal during the algorithm. Man $B_{\tau(n)}$ is paired with his first choice, woman $b_{\tau(n)}$. Man $B_{\tau(n-1)}$ proposes to woman $b_{\tau(n)}$ and gets rejected as woman $b_{\tau(n)}$ prefers man $B_{\tau(n)}$ over $B_{\tau(n-1)}$. Man $B_{\tau(n-1)}$ then proposes to woman $b_{\tau(n-1)}$ and gets accepted. In this manner, man $B_{\tau(i)}$'s proposals to women $b_{\tau(n)}, b_{\tau(n-1)}, \ldots, b_{\tau(i+1)}$ are all rejected as woman $b_{\tau(j)}$, $i+1 \le j \le n$, prefers man $B_{\tau(j)}$ over man $B_{\tau(i)}$. Hence, $B_{\tau(i)}$ is paired up with woman $b_{\tau(i)}$ for $1 \le i \le n$. Therefore, the male-optimal matching matches men A_i, C_i and B_i with women a_i, c_i and b_i for $1 \le i \le n$.

We find the female-optimal matching by reversing the roles of men and women. In other words, women make proposals and men accept or reject them. Suppose the women propose in the order $\{b_1, \ldots, b_n, c_1, \ldots, c_n, a_{\sigma n}, \ldots, a_{\sigma 1}\}$. Women b_i and c_i are paired up with their first choices, namely, men $A_{\rho^{-1} i}$ and B_i. Woman $a_{\sigma n}$ is paired with her first choice, namely, man C_n. Woman $a_{\sigma(n-1)}$ proposes to man C_n and gets rejected as man C_n prefers woman $a_{\sigma n}$ over $a_{\sigma(n-1)}$. Woman $a_{\sigma(n-1)}$ then proposes to man C_{n-1} and gets accepted. In this manner, woman $a_{\sigma i}$'s proposals to men $C_n, C_{n-1}, \ldots, C_{i+1}$ are all rejected as man C_j, $i+1 \le j \le n$, prefers woman $a_{\sigma j}$ over woman $a_{\sigma i}$. Hence, $a_{\sigma i}$ is paired up with man C_i for $1 \le i \le n$. Therefore, the female-optimal matching matches women b_i, c_i and a_i with men $A_{\rho^{-1} i}$, B_i and $C_{\sigma^{-1} i}$, respectively, for $1 \le i \le n$.

As is always the case, as we move from the male-optimal to the female-optimal matching (by performing a sequence of rotations), the men go down their preference lists starting from their male-optimal matching partner (their best possible partner) and ending at their female-optimal matching partner (their worst possible partner) while the women go up their preference lists starting from their male-optimal matching partner (their worst) and ending at their female-optimal matching partner (their best). Hence, a man will never be paired with a woman who appears either ahead of his male-optimal matching partner or after his female-optimal matching partner on his preference list. Similarly, in any stable matching a woman will never be paired with a man who appears either ahead of her female-optimal matching partner or after her male-optimal matching partner on her preference list. Hence, the only part of a man's preference list that we need to consider is the sub-list that starts at the male-optimal matching partner and ends at the female-optimal matching partner. Similarly, for the women we need to consider the sub-list starting at the female-optimal matching partner and ending at the male-optimal matching partner. These sub-lists are typically referred to as their *truncated preference lists*, and these are as follows:

$$\begin{aligned}
&\text{For } e_i \in Rep(\sigma), \quad 0 \le m \le p_i - 1, \quad 1 \le j \le n, \\
A_{\sigma^m e_i} \quad &: \quad a_{\sigma^m e_i} b_{\rho\sigma^m e_i}, \\
C_{\sigma^{(m-1)} e_i} \quad &: \quad c_{\sigma^{(m-1)} e_i} a_{\sigma^m e_i}, \\
B_{\tau(j)} \quad &: \quad b_{\tau(j)} b_{\tau(j-1)} \cdots b_{\tau(1)} a_{\tau(j)} c_{\tau(j)}, \quad \tau(j) \ne \sigma^{p_i - 1} e_i \\
B_{\tau(j)} \quad &: \quad b_{\tau(j)} b_{\tau(j-1)} \cdots b_{\tau(1)} a_{\tau(j)} c_{\sigma^{-1}\tau(j)} a_{\sigma^{-1}\tau(j)} \cdots \\
&\qquad c_{\sigma\tau(j)} a_{\sigma\tau(j)} c_{\tau(j)}, \quad \tau(j) = \sigma^{p_i - 1} e_i
\end{aligned}$$

$$\begin{aligned}
&\text{For } f_i \in Rep(\rho), \\
b_{\rho^m f_i} \quad &: \quad A_{\rho^{(m-1)} f_i} B_{\rho^m f_i}, \quad 0 \le m \le q_i - 1 \\
c_{\rho^m f_i} \quad &: \quad B_{\rho^m f_i} C_{\rho^m f_i}, \quad 0 \le m \le q_i - 1 \\
a_{\rho^m f_i} \quad &: \quad C_{\sigma^{-1}(\rho^m f_i)} C_{\sigma^{-1}(\rho^m f_i)-1} \cdots C_1 B_{\rho^m f_i} A_{\rho^m f_i}, \quad 0 \le m \le q_i - 2 \\
a_{\rho^{(q_i-1)} f_i} \quad &: \quad C_{\sigma^{-1}(\rho^{(q_i-1)} f_i)} C_{\sigma^{-1}(\rho^{(q_i-1)} f_i)-1} \cdots C_1 \\
&\qquad B_{\rho^{(q_i-1)} f_i} A_{\rho^{(q_i-2)} f_i} B_{\rho^{(q_i-2)} f_i} \cdots B_{\rho^2 f_i} A_{\rho f_i} B_{\rho f_i} A_{f_i} B_{f_i} A_{\rho^{(q_i-1)} f_i}.
\end{aligned}$$

4.3. Extracting rotations

We first observe that the male-optimal matching and female-optimal matching partners are different for every person. This implies that each man is involved in at least one rotation and, hence, every man has a well-defined suitor with respect to the male-optimal stable matching. Also, every man has at least two stable partners. The truncated preference lists of men A_i and C_i, $1 \leq i \leq n$, are of length two each. Hence, men A_i and C_i are each involved in exactly one rotation. Their suitors in the male-optimal matching $\mathcal{M}_0$ are $S_{\mathcal{M}_0}(A_i) = b_{\rho i}$ and $S_{\mathcal{M}_0}(C_i) = a_{\sigma i}$, respectively.

Note: Throughout this section we use $\mathcal{M}_0$ to denote the male-optimal stable matching.

Lemma 8. *In a stable matching $\mathcal{M}$, if A_i is paired with a_i, then $S_{\mathcal{M}}(A_i) = b_{\rho i} = S_{\mathcal{M}_0}(A_i)$.*

Proof. The truncated preference lists of man A_i and woman $b_{\rho i}$ are $a_i b_{\rho i}$ and $A_i B_{\rho i}$, respectively. Since $b_{\rho i}$ is paired with $B_{\rho i}$ in the male-optimal stable matching $\mathcal{M}_0$, the spouse of $b_{\rho i}$ in $\mathcal{M}$ is a man $M^* \in \{A_i, B_{\rho i}\}$. Since A_i is paired with a_i in $\mathcal{M}$, $b_{\rho i}$ is paired with $B_{\rho i}$. Hence, $b_{\rho i}$ prefers A_i over her partner in $\mathcal{M}$. This, in turn, implies that the suitor of A_i in $\mathcal{M}$, $S_{\mathcal{M}}(A_i)$, is $b_{\rho i}$. □

Lemma 9. *In a stable matching $\mathcal{M}$, if C_i is paired with c_i, then $S_{\mathcal{M}}(C_i) = a_{\sigma i} = S_{\mathcal{M}_0}(C_i)$.*

Proof. The truncated preference list of man C_i is $c_i a_{\sigma i}$. The truncated preference list of woman $a_{\sigma i}$ is either

$$C_i C_{i-1} \cdots C_1 B_{\sigma i} A_{\sigma i} \quad \text{or} \quad C_i C_{i-1} \cdots C_1 B_{\sigma i} A_{\rho^{-1}\sigma i} B_{\rho^{-1}\sigma i} \cdots A_{\rho\sigma i} B_{\rho\sigma i} A_{\sigma i}.$$

We note that the truncated list of $a_{\sigma i}$ starts with C_i. In $\mathcal{M}_0$, $a_{\sigma i}$ is paired up with $A_{\sigma i}$. This implies that in the current stable matching $\mathcal{M}$, $a_{\sigma i}$ is paired with a man M^* who is as high as $A_{\sigma i}$ on her preference list. As C_i is paired up with c_i in $\mathcal{M}$, $M^* \neq C_i$. Hence, in the current stable matching $\mathcal{M}$, $a_{\sigma i}$ prefers C_i over M^*. This, in turn, implies that $S_{\mathcal{M}}(C_i) = a_{\sigma i}$. □

Next we prove that $S_{\mathcal{M}_0}(B_i) = a_i$.

Lemma 10. *The suitor of man $B_{\tau(i)}$ in $\mathcal{M}_0$ is $S_{\mathcal{M}_0}(B_{\tau(i)}) = a_{\tau(i)}$.*

Proof. The truncated preference list of man $B_{\tau(i)}$ depends on the position of his subscript in the ρ-cycle. But we note that the initial part of the truncated preference list of $B_{\tau(i)}$ is $b_{\tau(i)} b_{\tau(i-1)} \cdots b_{\tau(1)} a_{\tau(i)}$ for all i. Since our arguments only require the initial part of the truncated preference list, we do not have to consider separate cases. The spouse of $B_{\tau(i)}$ in $\mathcal{M}_0$ is $sp_{\mathcal{M}_0}(B_{\tau(i)}) = b_{\tau(i)}$. Suppose the suitor of $B_{\tau(i)}$ is $S_{\mathcal{M}_0}(B_{\tau(i)}) = b_{\tau(j)}$ for some j, $1 \leq j < i$. This would imply that $b_{\tau(j)}$ prefers $B_{\tau(i)}$ over $B_{\tau(j)}$. But the initial part of $b_{\tau(j)}$'s preference list is $A_{\rho^{-1}\tau(j)} B_{\tau(j)} \cdots$. Hence, $b_{\tau(j)}$ prefers $B_{\tau(j)}$ over $B_{\tau(i)}$. This contradicts the assumption that $S_{\mathcal{M}_0}(B_{\tau(i)}) = b_{\tau(j)}$. This would imply that $S_{\mathcal{M}_0}(B_{\tau(i)}) \neq b_{\tau(j)}$ for every $j < i$. Since $a_{\tau(i)}$ is paired up with $A_{\tau(i)}$ in the male-optimal matching $\mathcal{M}_0$ and $a_{\tau(i)}$ prefers $B_{\tau(i)}$ over $A_{\tau(i)}$, $S_{\mathcal{M}_0}(B_{\tau(i)}) = a_{\tau(i)}$. □

The next two lemmas give the suitor of B_i in stable matchings which satisfy certain conditions.

Lemma 11. *In a stable matching $\mathcal{M}$, if C_k is paired with c_k for $1 \leq k \leq n$ and $B_{\tau(i)}$ is paired with $b_{\tau(i)}$, then $S_{\mathcal{M}}(B_{\tau(i)}) = a_{\tau(i)} = S_{\mathcal{M}_0}(B_{\tau(i)})$.*

Proof. The initial part of the truncated preference list of man $B_{\tau(i)}$ is $b_{\tau(i)} b_{\tau(i-1)} \cdots b_{\tau(1)}\ a_{\tau(i)}$. The spouse of $B_{\tau(i)}$ in $\mathcal{M}$ is $sp_{\mathcal{M}}(B_{\tau(i)}) = b_{\tau(i)}$. Suppose the suitor of $B_{\tau(i)}$ is $S_{\mathcal{M}}(B_{\tau(i)}) = b_{\tau(j)}$ for some j, $1 \leq j < i$. This would imply that $b_{\tau(j)}$ prefers $B_{\tau(i)}$ over $B_{\tau(j)}$. As the initial part of $b_{\tau(j)}$'s preference list is $A_{\rho^{-1}\tau(j)} B_{\tau(j)} \cdots$ and $b_{\tau(j)}$ is paired with $B_{\tau(j)}$ in the male-optimal stable matching, the partner of $b_{\tau(j)}$ in the current matching $\mathcal{M}$ would be a man M^* who is as high as $B_{\tau(j)}$ on her preference list. Since $b_{\tau(j)}$ prefers $B_{\tau(j)}$ over $B_{\tau(i)}$, $b_{\tau(j)}$ would prefer M^* over $B_{\tau(i)}$. This contradicts the assumption that $S_{\mathcal{M}}(B_{\tau(i)}) = b_{\tau(j)}$. This would entail that $S_{\mathcal{M}}(B_{\tau(i)}) \neq b_{\tau(j)}$ for every $j < i$.

Next we will show that $S_{\mathcal{M}}(B_{\tau(i)}) = a_{\tau(i)}$. In the male-optimal stable matching $\mathcal{M}_0$, $a_{\tau(i)}$ is paired up with $A_{\tau(i)}$. This implies that in the current stable matching $\mathcal{M}$, $a_{\tau(i)}$ is paired with a man M^* who is as high as $A_{\tau(i)}$ on her preference list, i.e. $sp_{\mathcal{M}}(a_{\tau(i)}) = M^*$. We note that $a_{\tau(i)}$'s truncated preference list is either

$$C_{\sigma^{-1}(\tau(i))} C_{\sigma^{-1}(\tau(i))-1} \cdots C_1 B_{\tau(i)} A_{\tau(i)} \quad \text{or} \quad C_{\sigma^{-1}(\tau(i))} C_{\sigma^{-1}(\tau(i))-1} \cdots C_1 B_{\tau(i)} A_{\rho^{-1}(\tau(i))} B_{\rho^{-1}(\tau(i))} \cdots A_{\rho\tau(i)} B_{\rho\tau(i)} A_{\tau(i)}.$$

The initial part of $a_{\tau(i)}$'s truncated list is $C_{\sigma^{-1}(\tau(i))} C_{\sigma^{-1}(\tau(i))-1} \cdots C_1 B_{\tau(i)}$. As C_k is paired up with c_k for $1 \leq k \leq n$, and $B_{\tau(i)}$ is paired up with $b_{\tau(i)}$ in the current matching $\mathcal{M}$, $M^* \notin \{C_{\sigma^{-1}(\tau(i))}, C_{\sigma^{-1}(\tau(i))-1}, \ldots C_1, B_{\tau(i)}\}$. Hence, in the current matching $\mathcal{M}$, $a_{\tau(i)}$ prefers $B_{\tau(i)}$ over her partner M^*. This, in turn, implies that $S_{\mathcal{M}}(B_{\tau(i)}) = a_{\tau(i)}$. □

Lemma 12. *In a stable matching $\mathcal{M}$, if, for all k, woman a_k is paired with man M_k, who is at least as high as B_k on her preference list, and if $B_{\tau(i)}$ is paired with $a_{\tau(i)}$, then $S_{\mathcal{M}}(B_{\tau(i)}) = c_{\tau(i)} = sp_{\mathcal{M}_t}(B_{\tau(i)})$, where $\mathcal{M}_t$ is the female-optimal stable matching.*

Proof. We will consider two cases depending on the preference list of $B_{\tau(i)}$.

Case (i) The truncated preference list of man $B_{\tau(i)}$ is

$$b_{\tau(i)} b_{\tau(i-1)} \cdots b_{\tau(1)} a_{\tau(i)} c_{\tau(i)}.$$

As the truncated preference list of woman $c_{\tau(i)}$ is $B_{\tau(i)} C_{\tau(i)}$ and the spouse of $c_{\tau(i)}$ in the male-optimal stable matching is $C_{\tau(i)}$, $sp_{\mathcal{M}}(c_{\tau(i)}) \in \{B_{\tau(i)}, C_{\tau(i)}\}$. As the spouse of $B_{\tau(i)}$ in $\mathcal{M}$ is $a_{\tau(i)}$ (by assumption), $sp_{\mathcal{M}}(c_{\tau(i)}) = C_{\tau(i)}$. Hence, $c_{\tau(i)}$ prefers $B_{\tau(i)}$ over her partner in $\mathcal{M}$. Therefore, the suitor of $B_{\tau(i)}$ in $\mathcal{M}$, $S_{\mathcal{M}}(B_{\tau(i)}) = c_{\tau(i)}$.

Case (ii) The truncated preference list of man $B_{\tau(i)}$ is

$$b_{\tau(i)}b_{\tau(i-1)}\cdots b_{\tau(1)}a_{\tau(i)}c_{\sigma^{-1}(\tau(i))}a_{\sigma^{-1}(\tau(i))}\cdots c_{\sigma\tau(i)}a_{\sigma\tau(i)}c_{\tau(i)}.$$

As the spouse of $B_{\tau(i)}$ in $\mathcal{M}$ is $a_{\tau(i)}$ (by assumption), and $c_{\tau(i)}$ is the partner of $B_{\tau(i)}$ in the female-optimal stable matching, $S_{\mathcal{M}}(B_{\tau(i)}) \in \{c_{\sigma^{-1}(\tau(i))}, a_{\sigma^{-1}(\tau(i))}, \ldots c_{\sigma\tau(i)}, a_{\sigma\tau(i)}, c_{\tau(i)}\}$.

Suppose $S_{\mathcal{M}}(B_{\tau(i)}) = c_{\sigma^k(\tau(i))} \neq c_{\tau(i)}$. As the initial part of the preference list of woman $c_{\sigma^k(\tau(i))}$ is $B_{\sigma^k(\tau(i))}C_{\sigma^k(\tau(i))}$, we see that $c_{\sigma^k(\tau(i))}$ prefers $C_{\sigma^k(\tau(i))}$ over $B_{\tau(i)}$. As the spouse of $c_{\sigma^k(\tau(i))}$ in the male-optimal stable matching is $C_{\sigma^k(\tau(i))}$, then $sp_{\mathcal{M}}(c_{\sigma^k(\tau(i))}) \stackrel{\text{def}}{=} M^*$ is at least as high as $C_{\sigma^k(\tau(i))}$ on her preference list. Hence, $c_{\sigma^k(\tau(i))}$ prefers her partner in $\mathcal{M}$ over $B_{\tau(i)}$. Therefore, $S_{\mathcal{M}}(B_{\tau(i)}) = c_{\sigma^k(\tau(i))} (\neq c_{\tau(i)})$ is not possible.

Suppose $S_{\mathcal{M}}(B_{\tau(i)}) = a_{\sigma^k(\tau(i))} \neq a_{\tau(i)}$. The initial part of the truncated preference list of woman $a_{\sigma^k(\tau(i))}$ is $C_{\sigma^{k-1}(\tau(i))}C_{\sigma^{k-1}(\tau(i))-1}\cdots C_1 B_{\sigma^k(\tau(i))}$. As the partner of $a_{\sigma^k(\tau(i))}$ in the stable matching $\mathcal{M}$ is at least as high as $B_{\sigma^k(\tau(i))}$, we have

$$M^* \in \{C_{\sigma^{k-1}(\tau(i))}, C_{\sigma^k(\tau(i))-1}, \ldots, C_1, B_{\sigma^k(\tau(i))}\}.$$

Hence, $a_{\sigma^k(\tau(i))}$ prefers M^* over $B_{\tau(i)}$. Therefore, $S_{\mathcal{M}}(B_{\tau(i)}) = a_{\sigma^k(\tau(i))} (\neq a_{\tau(i)})$ is not possible.

Now we will show that $S_{\mathcal{M}}(B_{\tau(i)}) = c_{\tau(i)}$. As the truncated preference list of woman $c_{\tau(i)}$ is $B_{\tau(i)}C_{\tau(i)}$ and the spouse of $c_{\tau(i)}$ in the male-optimal stable matching is $C_{\tau(i)}$, $sp_{\mathcal{M}}(c_{\tau(i)}) \in \{B_{\tau(i)}, C_{\tau(i)}\}$. As the spouse of $B_{\tau(i)}$ in $\mathcal{M}$ is $a_{\tau(i)}$, $sp_{\mathcal{M}}(c_{\tau(i)}) = C_{\tau(i)}$. Hence, $c_{\tau(i)}$ prefers $B_{\tau(i)}$ over her partner in $\mathcal{M}$. Therefore, the suitor of $B_{\tau(i)}$ in $\mathcal{M}$, $S_{\mathcal{M}}(B_{\tau(i)}) = c_{\tau(i)}$. □

We will later observe that a stable matching obtained after performing a set of ρ-rotations satisfies conditions laid out in Lemmas 8 and 11. Hence, the above lemmas help in establishing the suitors of A-men and B-men in the stable matchings obtained after performing ρ-rotations.

We note that the **Find-All-Rotations** algorithm obtains all the rotations of the instance irrespective of whatever ordering of the men we use in that procedure (to initialize the first proposal to his suitor). We order the men as follows: $\{A_1, \ldots, A_n, C_1, \ldots, C_n, B_1, \ldots, B_n\}$. In the male-optimal matching $\mathcal{M}_0$, A_1 is paired with a_1. **Find-All-Rotations** starts with man A_1 whose suitor is $b_{\rho 1}$. The sequence in that algorithm starts with the pair (A_1, a_1). As $b_{\rho 1}$ is the suitor of A_1, the next pair in the sequence is $(sp_{\mathcal{M}_0}(b_{\rho 1}), b_{\rho 1}) = (B_{\rho 1}, b_{\rho 1})$. With $a_{\rho 1}$ being the suitor of $B_{\rho 1}$, the next pair is $(sp_{\mathcal{M}_0}(a_{\rho 1}), a_{\rho 1}) = (A_{\rho 1}, a_{\rho 1})$. The pair $(A_{\rho 1}, a_{\rho 1})$ results in $(B_{\rho^2 1}, b_{\rho^2 1})$. In this manner, we grow the sequence (A_1, a_1), $(B_{\rho 1}, b_{\rho 1})$, $(A_{\rho 1}, a_{\rho 1})$, $(B_{\rho 2}, b_{\rho 2})$, ... As the suitor of A_i is $b_{\rho i}$ and that of B_i is a_i, we observe that the sequence alternates between A-men and B-men. We also note that the pair (A_1, a_1) results in the pair $(A_{\rho 1}, a_{\rho 1})$ and the pair $(B_{\rho 1}, b_{\rho 1})$ results in $(B_{\rho^2 1}, a_{\rho^2 1})$. In other words, the subscripts of the A-men and the B-men involved in the above sequence are from a ρ-cycle, in particular, the ρ-cycle containing 1. Suppose the ρ-cycle containing 1 is of size p_1, that is, the ρ-cycle containing 1 is $(1, 2, \ldots, p_1)$. Then the sequence we end up with is

$$\begin{aligned}&\{(A_1, a_1), (B_{\rho 1}, b_{\rho 1}), (A_{\rho 1}, a_{\rho 1}), \ldots, (B_{\rho^{p_1-1}1}, b_{\rho^{p_1-1}1}), (A_{\rho^{p_1-1}1}, a_{\rho^{p_1-1}1}), (B_{\rho^{p_1}1}, b_{\rho^{p_1}1})\}\\ &\quad = \{(A_1, a_1), (B_2, b_2), (A_2, a_2), \ldots, (B_{p_1}, b_{p_1}), (A_{p_1}, a_{p_1}), (B_1, b_1)\},\end{aligned}$$

using that $(B_{\rho^{p_1}1}, b_{\rho^{p_1}1}) = (B_1, b_1)$. Lemma 11 tells us this ends the sequence in the **Find-All-Rotations** algorithm, and we have therefore found a rotation.

(Note: If we start with any A_i or B_i, with $i \in (1, 2, \ldots, p_1)$, we will discover the same rotation, as the resulting sequence we find is a cyclic shift of the one given above.)

After applying the above ρ-rotation, A_i is paired with $b_{\rho i}$ (for $1 \leq i \leq p_1$), his partner in the female-optimal matching. Hence, in this new stable matching, the men $A_1, \ldots, A_{p_1}$ do not have suitors and will therefore not participate in future rotations. We also note that the only men who changed their partners in the above rotation were A-men and B-men with subscripts in the ρ-cycle containing 1. Hence, C_k is still paired up with c_k for $1 \leq k \leq n$, and A_i and B_i are paired up with a_i and b_i, respectively, when the subscript i does *not* belong to the ρ-cycle containing 1.

Let $\mathcal{M}_1$ denote this new matching after applying this first ρ-cycle we have discovered. We note that $\mathcal{M}_1$ satisfies the conditions laid out in Lemmas 8 and 11, which, in turn, tells us that $S_{\mathcal{M}_1}(A_i) = b_{\rho i}$ and $S_{\mathcal{M}_1}(B_i) = a_i$ for $i \notin \{1, 2, \ldots, p_1\}$. **Find-All-Rotations** then picks the next man who has a well-defined suitor, namely A_{p_1+1}, whose suitor is $S_{\mathcal{M}_1}(A_{p_1+1}) = b_{\rho(p_1+1)}$, and constructs the next rotation. From the above exercise of constructing a rotation corresponding to a ρ-cycle, it is clear that the rotation containing man A_{p_1+1} will be a ρ-rotation involving A-men and B-men whose subscripts belong to the ρ-cycle that contains p_1+1. Proceeding in this manner, we obtain all rotations (ρ-rotations) involving men A_i, $1 \leq i \leq n$.

Every B_i will participate in exactly one ρ-rotation as every $i \in [n]$, belongs to exactly one cycle of the permutation ρ. After applying all the ρ-rotations, we will obtain a stable matching, say $\mathcal{M}'$, in which the spouses of men A_i, B_i and C_i are $b_{\rho i}$, a_i, and c_i, respectively. As was observed before, all the A-men are paired up with their partners from the female-optimal stable matching. Thus, none of the A-men will participate in any of the future rotations. As was also noted, the C-men each participate in exactly one rotation and the suitor of C_i, $1 \leq i \leq n$, is $a_{\sigma i}$ as long as C_i is paired with c_i in the stable matching. From Lemma 12, it follows that the suitor of B_i in $\mathcal{M}'$ is c_i. All of the B-men and C-men have well-defined suitors.

The next man picked by **Find-All-Rotations** is C_1 whose suitor is $a_{\sigma i}$. The sequence starts with the pair (C_i, c_i). As $a_{\sigma i}$ is the suitor of C_i, the next pair in the sequence is $(sp_{\mathcal{M}'}(a_{\sigma i}), a_{\sigma i}) = (B_{\sigma i}, a_{\sigma i})$. Similarly, as the suitor of $B_{\sigma i}$ is $c_{\sigma i}$, the next pair in the sequence is $(sp_{\mathcal{M}'}(c_{\sigma i}), c_{\sigma i}) = (C_{\sigma i}, c_{\sigma i})$. Continuing from $(C_{\sigma i}, c_{\sigma i})$, we get the pair $(B_{\sigma^2 i}, a_{\sigma^2 i})$. Proceeding in this manner, we generate the rotation. We note that the suitor of B_i is c_i and that of C_i is $a_{\sigma i}$, thereby forcing us to alternate between C-men and B-men. We also note that the pair (C_i, c_i) eventually results in the pair $(C_{\sigma i}, c_{\sigma i})$ and $(B_{\sigma i}, a_{\sigma i})$ resulted in the pair $(B_{\sigma^2 i}, a_{\sigma^2 i})$. In other words, the subscripts of the C-men and the B-men in the rotation are governed by a σ-cycle, in particular, the σ-cycle containing 1. Suppose the σ-cycle containing 1 is of size q_1, that is, the σ-cycle containing 1 is $(1, \sigma 1, \dots, \sigma^{q_1-1}1)$. Then the rotation we end up with is $\{(C_1, c_1), (B_{\sigma 1}, a_{\sigma 1}), (C_{\sigma 1}, c_{\sigma 1}), \dots, (B_{\sigma^{q_1-1}1}, a_{\sigma^{q_1-1}1}), (C_{\sigma^{q_1-1}1}, c_{\sigma^{q_1-1}1}), (B_{\sigma^{q_1}1}, a_{\sigma^{q_1}1})\}$, where $(B_{\sigma^{q_1}1}, a_{\sigma^{q_1}1}) = (B_1, a_1)$.

After performing the above σ-rotation, for $0 \le k \le q_1 - 1$, $C_{\sigma^k 1}$, is paired with $a_{\sigma^{k+1}1}$, his partner in the female-optimal matching. Hence, men $C_1, \dots, C_{\sigma^{q_1-1}1}$ no longer have suitors and will not participate in any future rotations. We note that the only men who changed their partners in the above rotation were C-men and B-men with subscripts in the σ-cycle containing 1. Hence, A_k is still paired with $b_{\rho k}$ for $k \in [n]$, and C_i and B_i are paired up with c_i and a_i, respectively, when the subscript i does not belong to the σ-cycle containing 1.

Let $\mathcal{M}'_1$ denote the new matching after applying this σ-rotation. We note that $\mathcal{M}'_1$ satisfies the conditions laid out in Lemmas 9 and 12, which, in turn, entails that $S_{\mathcal{M}'_1}(C_i) = a_{\sigma i}$ and $S_{\mathcal{M}'_1}(B_i) = c_i$ for $i \notin \{1, \sigma 1, \dots, \sigma^{q_1-1}1\}$. **Find-All-Rotations** picks the next man who has a well-defined suitor, say C_{i_1}, where

$$i_1 = \min\{i : C_i \text{ is paired with } c_i \text{ in } \mathcal{M}'_1 \text{ and } 1 \le i \le n\},$$

and constructs a new rotation. The suitor of C_{i_1} is $S_{\mathcal{M}'_1}(C_{i_1}) = a_{\sigma i_1}$. From the above exercise of constructing a rotation corresponding to a σ-cycle, it is clear that the rotation containing man C_{i_1} will be a σ-rotation involving C-men and B-men whose subscripts belong to the σ-cycle containing i_1. Proceeding in this manner, we obtain all σ-rotations involving men C_i, $1 \le i \le n$. Each B_i will participate in exactly one σ-rotation, as every $i \in [n]$ belongs to exactly one cycle of the permutation σ.

After applying all the σ-rotations, we have a stable matching, say $\mathcal{M}''$, in which the spouses of men A_i, B_i and C_i are $b_{\rho i}$, c_i, and $a_{\sigma i}$, respectively. All the men are paired up with their partners from the female-optimal stable matching. Hence, $\mathcal{M}'' = \mathcal{M}_t$, where $\mathcal{M}_t$ is the female-optimal stable matching. Therefore we do not have any further rotations to extract. Hence, the only rotations in the rotation poset of the stable matching instance are the rotations governed by the ρ- and σ-cycles, namely ρ-rotations and σ-rotations. Therefore, the only stable pairs of the instance are (A_i, a_i), $(A_i, b_{\rho i})$, (B_i, b_i), (B_i, a_i), (B_i, c_i), (C_i, c_i), and $(C_i, a_{\sigma i})$ for $i \in [n]$.

We still have to prove that the ρ-rotations correspond to vertices in V_1 and the σ-rotations correspond to vertices in V_2. We prove this fact in the next section.

4.4. Ordering rotations

In this section, we compare rotations using the explicitly precedes relation as in Definition 3.6.

Recalling Definition 3.5, it follows that a man-woman pair (M, w) can be eliminated if and only if there exist stable pairs (M_1, w) and (M_2, w) such that w prefers M_1 over M and M appears as high as M_2 on w's preference list. In other words, a man-woman pair (M, w) can be eliminated if and only if M appears on the truncated preference list of w and is not the partner of w in the female-optimal stable matching. This identifies all the man-woman pairs eliminated by rotations of the instance.

Recall from the previous section that the only stable pairs of the matching instance are (A_i, a_i), $(A_i, b_{\rho i})$, (B_i, b_i), (B_i, a_i), (B_i, c_i), (C_i, c_i), $(C_i, a_{\sigma i})$ for $i \in [n]$. Of these stable pairs, $(A_i, b_{\rho i})$, (B_i, c_i) and $(C_i, a_{\sigma i})$ are pairs in the female-optimal stable matching. Hence, the only stable pairs that are eliminated by rotations are (A_i, a_i), (B_i, b_i), (B_i, a_i) and (C_i, c_i) for $i \in [n]$. We list all the eliminated pairs below and highlight those that are stable.

$$\begin{array}{rcl}
 & & \text{For } f_i \in Rep(\rho), \\
b_{\rho^m f_i} & : & \mathbf{B}_{\rho^{\mathbf{m}}\mathbf{f_i}}, \quad 0 \le m \le q_i - 1 \\
c_{\rho^m f_i} & : & \mathbf{C}_{\rho^{\mathbf{m}}\mathbf{f_i}}, \quad 0 \le m \le q_i - 1 \\
a_{\rho^m f_i} & : & C_{\sigma^{-1}(\rho^m f_i - 1)} \cdots C_1 \mathbf{B}_{\rho^{\mathbf{m}}\mathbf{f_i}} \mathbf{A}_{\rho^{\mathbf{m}}\mathbf{f_i}}, \quad 0 \le m \le q_i - 2 \\
a_{\rho^{(q_i-1)} f_i} & : & C_{\sigma^{-1}(\rho^{(q_i-1)} f_i - 1)} \cdots C_{\sigma^{-1}1} \mathbf{B}_{\rho^{(\mathbf{q_i}-\mathbf{1})}\mathbf{f_i}} A_{\rho^{(q_i-2)} f_i} B_{\rho^{(q_i-2)} f_i} \cdots B_{\rho^2 f_i} A_{\rho f_i} B_{\rho f_i} \mathbf{A}_{\rho^{(\mathbf{q_i}-\mathbf{1})}\mathbf{f_i}}
\end{array}$$

In Definition 3.6, rotation R eliminates pair (M, w) and rotation R' moves man M to woman w' such that M prefers w over w'. Hence, woman w and man M belong to rotations R and R', respectively.

Lemma 13. *Suppose R' is a ρ-rotation. Then there does not exist a rotation R which explicitly precedes R'. Therefore, every ρ-rotation is a minimal element of the rotation poset.*

Proof. Suppose there exists a rotation R which explicitly precedes

$$R' = \{(B_j, b_j), (A_j, a_j), (B_{j+1}, b_{j+1}), (A_{j+1}, a_{j+1}), \dots, (B_{j+q-1}, b_{j+q-1}), (A_{j+q-1}, a_{j+q-1})\}.$$

We consider two cases — (I) R is a ρ-rotation, (II) R is a σ-rotation.

Case (I) Suppose R is a ρ-rotation where

$$R = \{(B_i, b_i), (A_i, a_i), (B_{i+1}, b_{i+1}), (A_{i+1}, a_{i+1}), \dots, (B_{i+p-1}, b_{i+p-1}), (A_{i+p-1}, a_{i+p-1})\}.$$

The ρ-cycles corresponding to rotations R and R' are $(i, i+1, \dots, i+p-1)$ and $(j, j+1, \dots, j+q-1)$. Since any two ρ-cycles are disjoint, the corresponding ρ-rotations are disjoint, i.e. ρ-rotations R and R' do not share either a man or a woman. Since R explicitly precedes R', there exists a man-woman pair (M, w) with M belonging to rotation R' and w belonging to rotation R such that R eliminates the pair (M, w) and R' moves M to a woman w' below w on his list. In other words, there exist

$$\text{a man } M \in \{B_j, B_{j+1}, \dots, B_{j+q-1}, A_j, A_{j+1}, \dots, A_{j+q-1}\}, \quad \text{and}$$
$$\text{a woman } w \in \{b_i, b_{i+1}, \dots, b_{i+p-1}, a_i, a_{i+1}, \dots, a_{i+p-1}\}$$

satisfying the above property. We consider a set of sub-cases, depending upon possible values of M and w.

Subcase (I-a) $(M, w) \in \{(B_x, b_y), (A_x, b_y)\}$. We note that $x \neq y$, and x and y are from different ρ-cycles. From the table of eliminated pairs, we note that the set of eliminated pairs involving woman b_y is $\{(B_y, b_y)\}$. Hence, $(M, w) \neq (B_x, b_y)$ and $(M, w) \neq (A_x, b_y)$.

Subcase (I-b) $(M, w) = (A_x, a_y)$. We again note that $x \neq y$, and x and y are from different ρ-cycles. We also note that woman a_y could have one of two possible preference lists, which is reflected in the table of eliminated pairs. After performing rotation R, woman a_y is paired up with B_y. Hence, the set of pairs eliminated by rotation R involving woman a_y could be either

$$S = \{(A_y, a_y)\} \quad \text{or}$$
$$T = \{(A_{\rho^{-1}y}, a_y), (B_{\rho^{-1}y}, a_y), \dots, (A_{\rho y}, a_y), (B_{\rho y}, a_y), (A_y, a_y)\}.$$

Since we are only interested in eliminated pairs that involve an A-man, we consider subsets of S and T containing A-men, which are $\{(A_y, a_y)\}$ and $\{(A_{\rho^{-1}y}, a_y), (A_{\rho^{-2}y}, a_y), \dots, (A_{\rho y}, a_y), (A_y, a_y)\}$, respectively. We note that every element of $\{\rho^{-1}y, \rho^{-2}y, \dots, \rho y, y\}$ belongs to the ρ-cycle containing y. Since $x \neq y$ and x and y are from different ρ-cycles, $(A_x, a_y) \notin \{(A_y, a_y)\}$ and $(A_x, a_y) \notin \{(A_{\rho^{-1}y}, a_y), (A_{\rho^{-2}y}, a_y), \dots, (A_{\rho y}, a_y), (A_y, a_y)\}$. Hence, $(M, w) \neq (A_x, a_y)$.

Subcase (I-c) $(M, w) = (B_x, a_y)$. As before, $x \neq y$ and x, and y are from different ρ-cycles. From the table of eliminated pairs, we note that the set of eliminated pairs involving woman a_y could be either

$$S = \{(A_y, a_y)\} \quad \text{or}$$
$$T = \{(A_{\rho^{-1}y}, a_y), (B_{\rho^{-1}y}, a_y), \dots, (A_{\rho y}, a_y), (B_{\rho y}, a_y), (A_y, a_y)\}.$$

Since we are only interested in eliminated pairs that involve a B-man, we consider subsets of S and T containing B-men, which are $\emptyset$ and $\{(B_y, a_y), (B_{\rho^{-1}y}, a_y), (B_{\rho^{-2}y}, a_y), \dots, (B_{\rho y}, a_y)\}$, respectively. We note that every element of $\{y, \rho^{-1}y, \rho^{-2}y, \dots, \rho y\}$ belongs to the ρ-cycle containing y. Since $x \neq y$ and x and y are from different ρ-cycles,

$$(B_x, a_y) \notin \{(B_y, a_y), (B_{\rho^{-1}y}, a_y), (B_{\rho^{-2}y}, a_y), \dots, (B_{\rho y}, a_y)\}$$

and $(B_x, a_y) \notin \emptyset$ (vacuously). Hence, $(M, w) \neq (B_x, a_y)$.

Therefore, if R explicitly precedes R', then R is not a ρ-rotation.

Case (II) Suppose R is a σ-rotation where

$$R = \{(B_i, a_i), (C_i, c_i), (B_{\sigma i}, a_{\sigma i}), (C_{\sigma i}, c_{\sigma i}), \dots, (B_{\sigma^{p-1}i}, a_{\sigma^{p-1}i}), (C_{\sigma^{p-1}i}, c_{\sigma^{p-1}i})\}.$$

The σ- and ρ-cycles corresponding to rotations R and R' are $\sigma_1 = (i, \sigma i, \dots, \sigma^{p-1}i)$ and $\rho_1 = (j, j+1, \dots, j+q-1)$, respectively. Since R explicitly precedes R', there exists a man-woman pair (M, w) with M belonging to rotation R' and w belonging to rotation R such that R eliminates the pair (M, w) and R' moves M to a woman w' below w on his list. In other words, there exist a man $M \in \{B_j, B_{j+1}, \dots, B_{j+q-1}, A_j, A_{j+1}, \dots, A_{j+q-1}\}$ and a woman $w \in \{a_i, a_{\sigma i}, \dots, a_{\sigma^{p-1}i}, c_i, c_{\sigma i}, \dots, c_{\sigma^{p-1}i}\}$ satisfying the above property. Once again, there are a set of sub-cases to consider.

Subcase (II-a) $(M, w) \in \{(B_x, c_y), (A_x, c_y)\}$. From the table of eliminated pairs, we note that the set of eliminated pairs involving woman c_y is $\{(C_y, c_y)\}$. Hence, $(M, w) \neq (B_x, c_y)$ and $(M, w) \neq (A_x, c_y)$.

Subcase (II-b) $(M, w) \in \{(B_x, a_y), (A_x, a_y)\}$. We note that after performing rotation R', woman a_y is paired up with B_y. Hence, the pair (B_y, a_y) cannot be an eliminated pair. This implies that when the eliminated pair (M, w) is of the form (B_x, a_y), the subscript x cannot assume the value y. We observe that even though woman a_y could have one of two possible preference lists, the initial part of her preference list stays the same. We note that before performing the rotation R woman a_y is paired up with man B_y. After performing the rotation R, woman a_y is paired up with $C_{\sigma^{-1}y}$. Hence, the pairs eliminated by rotation R that involve woman a_y are $\{(C_{\sigma^{-1}(y)-1}, a_y), (C_{\sigma^{-1}(y)-2}, a_y), \dots, (C_1, a_y), (B_y, a_y)\}$. Since

$$\{(B_x, a_y), (A_x, a_y)\} \cap \{(C_{\sigma^{-1}(y)-1}, a_y), (C_{\sigma^{-1}(y)-2}, a_y), \dots, (C_1, a_y), (B_y, a_y)\} = \emptyset,$$

we see $(M, w) \neq (B_x, a_y)$ and $(M, w) \neq (A_x, a_y)$.

Therefore, if R explicitly precedes R', then R cannot be a σ-rotation.

From cases (I) and (II), we conclude that a ρ-rotation cannot be explicitly preceded either by another ρ-rotation or a σ-rotation. Therefore, every ρ-rotation is a minimal element in the rotation poset. $\square$

Lemma 14. *Suppose R is a σ-rotation. Then there does not exist a rotation R' such that R explicitly precedes R'. Therefore, every σ-rotation is a maximal element of the rotation poset.*

Proof. Suppose there exists a rotation R which explicitly precedes R'. We consider two cases.

Case (I) R' is a ρ-rotation. This case has been dealt with in case (II) of Lemma 13.

Case(II) R' is a σ-rotation. Let

$$R = \{(B_i, a_i), (C_i, c_i), (B_{\sigma i}, a_{\sigma i}), (C_{\sigma i}, c_{\sigma i}), \dots, (B_{\sigma^{p-1} i}, a_{\sigma^{p-1} i})(C_{\sigma^{p-1} i}, c_{\sigma^{p-1} i})\},$$

and

$$R' = \{(B_j, a_j), (C_j, c_j), (B_{\sigma j}, a_{\sigma j}), (C_{\sigma j}, c_{\sigma j}), \dots, (B_{\sigma^{q-1} j}, a_{\sigma^{q-1} j}), (C_{\sigma^{q-1} j}, c_{\sigma^{q-1} j})\}.$$

The σ-cycles corresponding to rotations R and R' are $(i, \sigma i, \dots, \sigma^{p-1} i)$ and $(j, \sigma j, \dots, \sigma^{q-1} j)$. Since any two σ-cycles are disjoint, the corresponding σ-rotations are disjoint, i.e. σ-rotations R and R' do not share either a man or a woman. As has been observed before, the implication of R explicitly preceding R' is that there exists a man–woman pair (M, w) with M belonging to rotation R' and w belonging to rotation R such that R eliminates the pair (M, w) and R' moves M to a woman w' below w on his list. In other words, there exist

$$\text{a man } M \in \{B_j, B_{\sigma j}, \dots, B_{\sigma^{q-1} j}, C_j, C_{\sigma j}, \dots, C_{\sigma^{q-1} j}\} \quad \text{and}$$
$$\text{a woman } w \in \{a_i, a_{\sigma i}, \dots, a_{\sigma^{p-1} i}, c_i, c_{\sigma i}, \dots, c_{\sigma^{p-1} i}\}$$

satisfying the above property. As the pair (M, w) has a set of possibilities, we consider a set of sub-cases.

Subcase (II-a) $(M, w) \in \{(B_x, c_y), (C_x, c_y)\}$. We note that $x \neq y$, and x and y are from different σ-cycles. From the table of eliminated pairs, we note that the set of eliminated pairs involving woman c_y is $\{(C_y, c_y)\}$. Hence, $(M, w) \neq (B_x, c_y)$ and $(M, w) \neq (C_x, c_y)$.

Subcase (II-b) $(M, w) = (B_x, a_y)$. As before, we note that $x \neq y$ and x and y are from different σ-cycles. We also note that the spouses of woman a_y before and after performing the rotation R are B_y and $C_{\sigma^{-1} y}$, respectively. Hence, the pairs eliminated by rotation R are

$$\{(C_{\sigma^{-1}(y)-1}, a_y), (C_{\sigma^{-1}(y)-2}, a_y), \dots, (C_1, a_y), (B_y, a_y)\}.$$

As $x \neq y$, $(B_x, a_y) \notin \{(C_{\sigma^{-1}(y-1)}, a_y), (C_{\sigma^{-1}(y-2)}, a_y), \dots, (C_{\sigma^{-1} 1}, a_y), (B_y, a_y)\}$. Therefore, $(M, w) \neq (B_x, a_y)$.

Subcase (II-c) $(M, w) = (C_x, a_y)$. We note that x and y are from different σ-cycles. We also note that the spouses of woman a_y before and after performing the rotation R are B_y and $C_{\sigma^{-1} y}$, respectively. Hence, the pairs eliminated by rotation R are

$$\{(C_{\sigma^{-1}(y)-1}, a_y), (C_{\sigma^{-1}(y)-2}, a_y), \dots, (C_1, a_y), (B_y, a_y)\}.$$

Suppose $(C_x, a_y) \in \{(C_{\sigma^{-1}(y)-1}, a_y), (C_{\sigma^{-1}(y)-2}, a_y), \dots, (C_1, a_y), (B_y, a_y)\}$. This implies $x \in \{\sigma^{-1}(y) - 1, \sigma^{-1}(y) - 2, \dots, 1\}$. Recalling that the only stable pairs are (A_i, a_i), $(A_i, b_{\rho i})$, (B_i, b_i), (B_i, a_i), (B_i, c_i), (C_i, c_i), $(C_i, a_{\sigma i})$ for $i \in [n]$, we see that (C_x, a_y) is an unstable pair as $x \neq \sigma^{-1} y$. Therefore, every pair of the form (C_x, a_y) that rotation R eliminates is an unstable pair. Since R explicitly precedes R' and the pairs of the form (C_x, a_y) eliminated by R are unstable, rotation R' has to move some C_x below a_y on his list. The initial part of the preference list of C_x is $c_x a_{\sigma x}$ for all $x \in \{1, 2, \dots, n\}$. In other words, C_x has $a_{\sigma x}$ above a_y for all $y \neq \sigma x$. After performing rotation R', C_x moves from c_x to $a_{\sigma x}$ for every $x \in \{j, \sigma j, \dots, \sigma^{q-1} j\}$. Therefore, rotation R' does not take C_x below a_y on his preference list for $x \in \{j, \sigma j, \dots, \sigma^{q-1} j\}$. Therefore, $(M, w) \neq (C_x, a_y)$.

Putting together cases (I) and (II), we conclude that if R is a σ-rotation, then it cannot explicitly precede any rotation R'. This, in turn, implies that every σ-rotation is a maximal element in the rotation poset. $\square$

From Lemmas 13 and 14, it follows that the rotation poset of our constructed matching instance is of height at most 1.

Lemma 15. *Suppose R is a ρ-rotation and R' is a σ-rotation. Then R explicitly precedes R' if and only if R and R' have a common man. In other words, the ρ and σ-cycles corresponding to R and R', respectively, have an element in common.*

Proof. Let

$$R = \{(B_i, b_i), (A_i, a_i), (B_{i+1}, b_{i+1}), (A_{i+1}, a_{i+1}), \dots, (B_{i+p-1}, b_{i+p-1}), (A_{i+p-1}, a_{i+p-1})\}$$

and

$$R' = \{(B_j, a_j), (C_j, c_j), (B_{\sigma j}, a_{\sigma j}), (C_{\sigma j}, c_{\sigma j}), \dots, (B_{\sigma^{q-1} j}, a_{\sigma^{q-1} j}), (C_{\sigma^{q-1} j}, c_{\sigma^{q-1} j})\}.$$

Suppose R and R' do not have a common man, i.e.

$$\{B_i, B_{i+1}, \dots, B_{i+p-1}\} \cap \{B_j, B_{\sigma j}, \dots, B_{\sigma^{q-1} j}\} = \emptyset.$$

This, in turn, entails that $\rho_1 \cap \sigma_1 = \emptyset$, where $\rho_1 = \{i, i+1, \dots, i+p-1\}$ and $\sigma_1 = \{j, \sigma j, \dots, \sigma^{q-1} j\}$. As has been observed before, the implication of R explicitly preceding R' is that there exists a man–woman pair (M, w) with M belonging to rotation R' and w belonging to rotation R such that R eliminates the pair (M, w) and R' moves M to a woman w' below w on his list. In other words, there exist

$$\text{a man } M \in \{B_j, B_{\sigma j}, \ldots, B_{\sigma^{q-1}j}, C_j, C_{\sigma j}, \ldots, C_{\sigma^{q-1}j}\} \quad \text{and}$$
$$\text{a woman } w \in \{b_i, b_{i+1}, \ldots, b_{i+p-1}, a_i, a_{i+1}, \ldots, a_{i+p-1}\}$$

satisfying the above property. We consider a set of sub-cases.

Case (I) $(M, w) \in \{(B_x, b_y), (C_x, b_y)\}$. We note that $x \neq y$, as $\rho_1 \cap \sigma_1 = \emptyset$ and $x \in \sigma_1$ and $y \in \rho_1$. From the table of eliminated pairs, we note that the set of eliminated pairs involving woman b_y is $\{(B_y, b_y)\}$. Hence, $(M, w) \notin \{(B_x, b_y), (C_x, b_y)\}$.

Case (II) $(M, w) \in \{(B_x, a_y), (C_x, a_y)\}$. We note that a_y could have one of two possible preference lists, which is reflected in the table of eliminated pairs.

The spouses of a_y before and after performing the rotation R are A_y and B_y, respectively. Hence, the set of pairs eliminated by rotation R involving woman a_y could be either $S = \{(A_y, a_y)\}$ or $T = \{(A_{\rho^{-1}y}, a_y), (B_{\rho^{-1}y}, a_y), \ldots, (A_{\rho y}, a_y), (B_{\rho y}, a_y), (A_y, a_y)\}$. Since none of the eliminated pairs involve a C-man, $(M, w) \neq (C_x, a_y)$.

With (C_x, a_y) eliminated from being a possible candidate, we are only interested in eliminated pairs that involve a B-man. We consider subsets of S and T containing B-men, which are $\emptyset$ and $\{(B_{\rho^{-1}y}, a_y), (B_{\rho^{-2}y}, a_y), \ldots, (B_{\rho y}, a_y)\}$, respectively. We note that every element of $\rho_1 = \{\rho^{-1}y, \rho^{-2}y, \ldots, \rho y, y\} = \{i, i+1, \ldots, i+p-1\}$ belongs to the ρ-cycle containing y. As σ_1 and ρ_1 do not have an element in common, $x \neq y$ and x does not belong to the ρ-cycle containing y. Therefore, $(B_x, a_y) \notin \{(B_{\rho^{-1}y}, a_y), (B_{\rho^{-2}y}, a_y), \ldots, (B_{\rho y}, a_y)\}$. Hence, $(M, w) \neq (B_x, a_y)$.

From cases (I) and (II), it follows that if R and R' do not share a common man, then there does not exist a man–woman pair (M, w) such that R eliminates (M, w) and R' moves M to w' below w. Hence, R does not explicitly precede R'.

Suppose rotations R and R' have a common man. In other words,

$$\{B_i, B_{i+1}, \ldots, B_{i+p-1}\} \cap \{B_j, B_{\sigma j}, \ldots, B_{\sigma^{q-1}j}\} \neq \emptyset.$$

This, in turn, entails that $\rho_1 \cap \sigma_1 \neq \emptyset$, where $\rho_1 = \{i, i+1, \ldots, i+p-1\}$ and $\sigma_1 = \{j, \sigma j, \ldots, \sigma^{q-1}j\}$. We also note that a ρ-cycle and a σ-cycle can have at most one element in common. Therefore, $\{B_i, B_{i+1}, \ldots, B_{i+p-1}\} \cap \{B_j, B_{\sigma j}, \ldots, B_{\sigma^{q-1}j}\} = \{B_l\}$, say.

As has been observed before, in order to establish that R explicitly precedes R', it is enough to produce a man–woman pair (M, w) with M belonging to rotation R' and w belonging to rotation R such that R eliminates the pair (M, w) and R' moves M to a woman w' below w on his list. In other words, it is enough to show that there exist a man $M \in \{B_j, B_{\sigma j}, \ldots, B_{\sigma^{q-1}j}, C_j, C_{\sigma j}, \ldots, C_{\sigma^{q-1}j}\}$ and a woman $w \in \{b_i, b_{i+1}, \ldots, b_{i+p-1}, a_i, a_{i+1} \ldots, a_{i+p-1}\}$ satisfying the above property. We show that the pair (B_l, b_l) is the required pair.

Since B_l participates in the rotation R, the spouses of B_l before and after the rotation R are b_l and a_l, respectively. Hence, the pair (B_l, b_l) is eliminated by R, and R moves B_l to a_l, which is below b_l. Since B_l belongs to R', the spouses of B_l before and after performing the rotation R' are a_l and c_l, respectively. Hence, R' moves B_l to c_l, which is below a_l. This entails that rotation R' moves B_l to c_l, which is below b_l on his preference list. Therefore, R eliminates the pair (B_l, b_l) and R' moves B_l to the woman c_l who is below a_l on B_l's preference list. This implies that R explicitly precedes R'.

Hence, it follows that if R and R' share a man, then R explicitly precedes R'. This proves the lemma. □

From Lemma 15, it follows that the rotation poset has an edge from a ρ-rotation to a σ-rotation (i.e. $\rho \leq \sigma$ in the ordering of the rotations) if and only if the rotations share a common man. Hence, the rotation poset has height 1. In other words, the rotation poset has an edge between two vertices if and only if the cycles corresponding to the vertices have an element in common. The edges of the bipartite graph, which was introduced early on, were defined in a similar fashion. Hence, the rotation poset when considered as a graph is isomorphic to a bipartite graph.

5. The 1-attribute case

In this section we concentrate our attention on the 1-attribute model. This case is very special and we establish the following result.

Theorem 2. *#SM(1-attribute) is solvable in polynomial time.*

Theorem 2 is obtained as a corollary of the following theorem.

Theorem 16. *In the 1-attribute model, the rotation poset of a stable matching instance is (isomorphic to) a path.*

Theorem 2 follows as it is straightforward to count the downsets of a path. We establish Theorem 16 through a series of lemmas.

First, we observe that in the 1-attribute model the men have only two possible preference lists for the women. The two preference lists are such that one is the reverse of the other. Similarly, the women have only two possible preference lists, one being the reverse of the other.

We start by establishing that every rotation in the 1-attribute model is of even size. In other words, every rotation involves an even number of men.

Lemma 17. *In the* 1*-attribute model, every rotation is of even size, and the preference lists of the men (and, similarly, the women) involved in the rotation alternate.*

Proof. We establish the statement by showing that two consecutive men in a rotation cannot both have the same preference list. Then, since there are only two possible preference lists and the preference lists of any two consecutive men have to be different, we conclude that the number of men involved in the rotation has to be even.

Suppose we have a rotation R of size k. Without loss of generality (by relabeling), we assume this rotation is $(B_0, b_0), (B_1, b_1), \ldots, (B_{k-1}, b_{k-1})$. Each man B_i is married to woman b_i (in M_1) before the rotation, and to woman $b_{i+1 \pmod k}$ (in M_2) after the rotation, as shown in the table below.

Men	(Before R) M_1	(After R) M_2
B_0	b_0	b_1
B_1	b_1	b_2
$\vdots$		
B_i	b_i	b_{i+1}
B_{i+1}	b_{i+1}	b_{i+2}
$\vdots$		
B_{k-2}	b_{k-2}	b_{k-1}
B_{k-1}	b_{k-1}	b_0

Before we proceed, we note that all subscripts that follow are computed $\mod k$.

To establish our result it is enough to show that two consecutive men in a rotation cannot both have the same preference lists.

So, suppose to the contrary that men B_i and B_{i+1} have the same preference list. Recalling that the rotation is female-improving, the preference list of B_i and B_{i+1} are shown below.

B_i	$\cdots$	b_i	$\cdots$	b_{i+1}
B_{i+1}	$\cdots$	b_{i+1}	$\cdots$	b_{i+2}

Since B_i and B_{i+1} have the same preference lists, b_i comes ahead of b_{i+1} on B_{i+1}'s preference list, as shown below.

B_i			$\cdots$	b_i	$\cdots$	b_{i+1}
B_{i+1}	$\cdots$	b_i	$\cdots$	b_{i+1}	$\cdots$	b_{i+2}

As M_1 is a stable matching, the pair (B_{i+1}, b_i) must not form a blocking pair to the stable pairs (B_i, b_i) and (B_{i+1}, b_{i+1}) in M_1. Since b_i comes ahead of b_{i+1} on B_{i+1}'s list, B_{i+1} must appear *after* B_i on b_i's preference list to ensure that (B_{i+1}, b_i) does not form such a blocking pair. Therefore, the preference lists for man B_{i+1} and woman b_i are as follows.

B_{i+1}	$\cdots$	b_i	$\cdots$	b_{i+1}	$\cdots$	b_{i+2}	b_i	$\cdots$	B_{i-1}	$\cdots$	B_i	$\cdots$	B_{i+1}

Comparing the preference lists of women b_i and b_{i+1},

b_i	$\cdots$	B_{i-1}	$\cdots$	B_i	$\cdots$	B_{i+1}
b_{i+1}	$\cdots$	B_i	$\cdots$	B_{i+1}	$\cdots$	

we note that they are the same. Hence, woman b_{i+1} also has B_{i-1} ahead of B_i on her list.

b_i			$\cdots$	B_{i-1}	$\cdots$	B_i	$\cdots$	B_{i+1}
b_{i+1}	$\cdots$	B_{i-1}	$\cdots$	B_i	$\cdots$	B_{i+1}	$\cdots$	

Because M_2 is also a stable matching and B_{i-1} is ahead of B_i on b_{i+1}'s preference list, b_{i+1} must appear *after* b_i on B_{i-1}'s preference list to prevent (B_{i-1}, b_{i+1}) from being a blocking pair in M_2. Comparing the preference lists of B_{i-1}, B_i and B_{i+1}, we note that

B_{i-1}			$\cdots$	b_{i-1}	$\cdots$	b_i	$\cdots$	b_{i+1}
B_i			$\cdots$	b_i	$\cdots$	b_{i+1}		
B_{i+1}	$\cdots$	b_i	$\cdots$	b_{i+1}	$\cdots$	b_{i+2}		

they are all the same.

We have shown that man B_{i-1} has the same preference list as men B_i and B_{i+1}, and that women b_i and b_{i+1} both have the same preference list. We can now repeat the argument with men B_{i-1} and B_i to conclude that men B_{i-2} and B_{i-1} have the same preference list and women b_{i-1} and b_i have the same preference list and so on. In this fashion, we can show that all men involved in the rotation have the same preference list and so do all the women involved.

We know that the men involved in a rotation get less happy with their partners as a result of applying the rotation. Since the men all have the same preference lists, the relative order of women b_1 through b_k should be the same. Suppose the order is $b_{i_1}, b_{i_2}, \ldots, b_{i_k}$. After the rotation, the man married to b_{i_1} would go down his list and the man married to b_{i_k} would go up his list, which cannot both happen at the same time. Hence, we cannot have a rotation if some two consecutive men on the rotation have the same preference lists.

Therefore, the preference lists of the men involved in the rotation have to alternate, forcing the rotation to be of even size. □

Lemma 18. *In the* 1*-attribute model, every rotation is of size* 2*.*

Proof. Suppose we have a rotation of size $2k$ involving men $B_0, B_1, \ldots, B_{2k-1}$ and women $b_0, b_1, \ldots, b_{2k-1}$, where $k > 1$. Every man B_i is married to woman b_i before the rotation and to woman $b_{i+1 \pmod k}$ after the rotation, as shown in the table below.

Men	Before	After
B_0	b_0	b_1
B_1	b_1	b_2
$\vdots$		
B_i	b_i	b_{i+1}
B_{i+1}	b_{i+1}	b_{i+2}
$\vdots$		
B_{2k-2}	b_{2k-2}	b_{2k-1}
B_{2k-1}	b_{2k-1}	b_0

Since men B_i with i even have the same preference lists, the relative order of women on their lists is the same. Suppose the order is $b_{i_1}, b_{i_2}, \ldots, b_{i_{2k}}$. Since, for $0 \le i \le k-1$, b_{2i} is ahead of b_{2i+1}, it is clear that i_1 is even.

Consider men B_{i_1-2}, B_{i_1-1}, and B_{i_1}. Note that we are implicitly using the assumption that $k > 1$ (otherwise there are not three distinct men). Their preference lists appear as follows.

B_{i_1-2}	$\cdots$	b_{i_1}	$\cdots$	b_{i_1-2}	$\cdots$	b_{i_1-1}
B_{i_1-1}			$\cdots$	b_{i_1-1}	$\cdots$	b_{i_1}
B_{i_1}			$\cdots$	b_{i_1}	$\cdots$	b_{i_1+1}

As all pairs (B_j, b_j) are part of a stable matching, the pair (B_{i_1-2}, b_{i_1}) should not form a blocking pair to the stable pairs (B_{i_1-2}, b_{i_1-2}) and (B_{i_1}, b_{i_1}). Since b_{i_1} comes ahead of b_{i_1-2} on B_{i_1-2}'s list, B_{i_1-2} should appear after B_{i_1} on b_{i_1}'s preference list to ensure that (B_{i_1-2}, b_{i_1}) does not form a blocking pair. The preference lists for women b_{i_1} and woman b_{i_1+1} are as follows.

b_{i_1}			$\cdots$	B_{i_1-1}	$\cdots$	B_{i_1}	$\cdots$	B_{i_1-2}	$\cdots$
b_{i_1+1}	$\cdots$	B_{i_1-2}	$\cdots$	B_{i_1}	$\cdots$	B_{i_1+1}	$\cdots$		

Note that B_{i_1-2} appears ahead of B_{i_1} on b_{i_1+1}'s list as the lists of b_{i_1} and b_{i_1+1} are reverses of each other.

Again, all pairs (B_j, b_{j+1}) are part of a stable matching and (B_{i_1-2}, b_{i_1+1}) could form a blocking pair to the stable pairs (B_{i_1-2}, b_{i_1-1}) and (B_{i_1}, b_{i_1+1}). Since B_{i_1-2} is ahead of B_{i_1} on b_{i_1+1}'s preference list, b_{i_1+1} has to appear after b_{i_1-1} on B_{i_1-2}'s preference list to prevent (B_{i_1-2}, b_{i_1+1}) from becoming a blocking pair. The preference lists of B_{i_1-2}, B_{i_1-1} and B_{i_1} are as follows.

B_{i_1-2}	$\cdots$	b_{i_1}	$\cdots$	b_{i_1-2}	$\cdots$	b_{i_1-1}	$\cdots$	b_{i_1+1}
B_{i_1-1}	$\cdots$	b_{i_1+1}	$\cdots$	b_{i_1-1}	$\cdots$	b_{i_1}	$\cdots$	
B_{i_1}			$\cdots$	b_{i_1}	$\cdots$	b_{i_1+1}	$\cdots$	

Note that b_{i_1+1} appears ahead of b_{i_1-1} on B_{i_1-1}'s list, as the lists of B_{i_1-2} and B_{i_1-1} are reverses of each other.

The pair (B_{i_1-1}, b_{i_1+1}) should not form a blocking pair to the stable pairs (B_{i_1-1}, b_{i_1-1}) and (B_{i_1+1}, b_{i_1+1}). Since b_{i_1+1} comes ahead of b_{i_1-1} on B_{i_1-1}'s list, B_{i_1-1} should appear after B_{i_1+1} on b_{i_1+1}'s preference list to ensure that (B_{i_1-1}, b_{i_1+1}) does not form a blocking pair. The preference lists for women b_{i_1} and woman b_{i_1+1} are as follows.

b_{i_1}			$\cdots$	B_{i_1-1}	$\cdots$	B_{i_1}	$\cdots$	B_{i_1-2}	$\cdots$
b_{i_1+1}	$\cdots$	B_{i_1-2}	$\cdots$	B_{i_1}	$\cdots$	B_{i_1+1}	$\cdots$	B_{i_1-1}	
b_{i_1+2}	$\cdots$	B_{i_1-1}	$\cdots$	B_{i_1+1}	$\cdots$	B_{i_1+2}	$\cdots$		

Note that B_{i_1-1} appears ahead of B_{i_1+1} on b_{i_1+2}'s list, as the lists of b_{i_1+1} and b_{i_1+2} are reverses of each other.

Similarly, (B_{i_1-1}, b_{i_1+2}) must not be a blocking pair to the stable pairs (B_{i_1-1}, b_{i_1}) and (B_{i_1+1}, b_{i_1+2}). Since B_{i_1-1} is ahead of B_{i_1+1} on b_{i_1+2}'s preference list, b_{i_1+2} has to appear after b_{i_1} on B_{i_1-1}'s preference list to prevent (B_{i_1-1}, b_{i_1+2}) from becoming a blocking pair. The preference lists of B_{i_1-2}, B_{i_1-1} and B_{i_1} are as follows.

B_{i_1-2}	$\cdots$	b_{i_1}	$\cdots$	b_{i_1-2}	$\cdots$	b_{i_1-1}	$\cdots$	b_{i_1+1}	
B_{i_1-1}	$\cdots$	b_{i_1+1}	$\cdots$	b_{i_1-1}	$\cdots$	b_{i_1}	$\cdots$	b_{i_1+2}	$\cdots$
B_{i_1}	$\cdots$	b_{i_1+2}	$\cdots$	b_{i_1}	$\cdots$	b_{i_1+1}	$\cdots$		

Note that b_{i_1+2} appears ahead of b_{i_1} on B_{i_1}'s list, as the lists of B_{i_1-1} and B_{i_1} are reverses of each other. This contradicts the relative order of the women on lists of men B_j with j even, since b_{i_1} should be first.

Hence, the size of any rotation is 2. □

Lemma 19. *In the* 1*-attribute model, every man (and woman) participates in at most one rotation.*

Proof. Suppose man B_1 participates in more than one rotation. Starting with his partner in the male-optimal matching, man B_1 goes down his preference list with each rotation he participates in. Suppose b_1 is the partner of B_1 in the male-optimal matching, and b_2 and b_3 are partners of B_1 after the first and second rotations, respectively, that involve B_1. Let B_2 and B_3 be the partners of b_2 and b_3 when they participate in the respective rotations with B_1. The preference lists of B_1, B_2, B_3, b_1, b_2, and b_3 are as follows.

B_1	$\cdots$	b_1	$\cdots$	b_2	$\cdots$	b_3	$\cdots$
B_2	$\cdots$	b_2	$\cdots$	b_1	$\cdots$		
B_3			$\cdots$	b_3	$\cdots$	b_2	$\cdots$

b_1			$\cdots$	B_2	$\cdots$	B_1	$\cdots$
b_2	$\cdots$	B_3	$\cdots$	B_1	$\cdots$	B_2	
b_3	$\cdots$	B_1	$\cdots$	B_3	$\cdots$		

We note that B_2 and B_3 have the same preference lists and B_1 has the reverse preference list. Hence, their preference lists appear as follows.

B_1			$\cdots$	b_1	$\cdots$	b_2	$\cdots$	b_3	$\cdots$	
B_2	$\cdots$	b_3	$\cdots$	b_2	$\cdots$	b_1	$\cdots$			
B_3					$\cdots$	b_3	$\cdots$	b_2	$\cdots$	b_1

Similarly, b_1 and b_3 have the same preference lists and b_2 has the reverse preference list. Hence, their preference lists are as follows.

b_1					$\cdots$	B_2	$\cdots$	B_1	$\cdots$	B_3
b_2			$\cdots$	B_3	$\cdots$	B_1	$\cdots$	B_2		
b_3	$\cdots$	B_2	$\cdots$	B_1	$\cdots$	B_3	$\cdots$			

When B_1 and B_2 participate in the rotation, their partners are b_1 and b_2 respectively. This implies that (B_2, b_2) is a stable pair and is part of a stable matching. Hence, the pair (B_2, b_3) cannot be a blocking pair. For (B_2, b_3) not to be a blocking pair, b_3 should be married to someone higher than B_2 on her list, say B_x. In other words, b_3 should be married to B_x before the rotation involving B_1 and B_2 occurs and cannot to married to anyone lower than B_x after the rotation has occurred because b_3 can only go up her preference list after future rotations.

b_3	$\cdots$	B_x	$\cdots$	B_2	$\cdots$	B_1	$\cdots$	B_3	$\cdots$

This implies that b_3 can never be married to B_1 or B_3 in the future, and the rotation involving the pairs (B_1, b_2) and (B_3, b_3) which happens after the rotation involving B_1 and B_2 violates that. Hence, any man (and woman) can participate in at most one rotation. □

We are now in a position to prove Theorem 16, which we repeat here.

Theorem 16. *In the* 1*-attribute model, the rotation poset of a stable matching instance is (isomorphic to) a path.*

Proof. In order to prove this theorem, we need to show that any two rotations are comparable, as this gives a total ordering on the set of rotations.

We start by computing the male-optimal and female-optimal stable matchings. The men who have the same partner in both matchings are removed along with their partners from the problem instance, as their presence or absence does not affect the rotation poset. So we may assume that every man and women in the stable matching instance is involved in at least one rotation.

Since every rotation involves exactly two men and two women (Lemma 18), and, by removing the men and women that are not involved in any rotations, we see that each man and woman that remains is involved in exactly one rotation (Lemma 19). Thus, the number of men and women in the (reduced) matching instance must be even.

Let the $2k$ men be denoted $\{B_1, \ldots, B_{2k}\}$ and the $2k$ women be denoted $\{b_1, \ldots, b_{2k}\}$. By relabeling, we can assume that the male-optimal matching pairs man B_i with woman b_i, and the female-optimal matching pairs man B_{2i-1} with woman b_{2i}, and man B_{2i} with woman b_{2i-1}. In other words, there are k rotations $R_1, R_2, \ldots, R_k$ and rotation R_i is of the form $\{(B_{2i-1}, b_{2i-1}), (B_{2i}, b_{2i})\}$. We want to show that any two rotations are comparable, i.e., for every $i, j \in \{1, 2, \ldots, k\}$, where $i \neq j$, either R_i precedes R_j or R_j precedes R_i.

Let us compare two rotations, say R_1 and R_2. The men and women involved in the two rotations are $\{B_1, B_2, B_3, B_4\}$ and $\{b_1, b_2, b_3, b_4\}$.

The preference list of B_2 is the reverse of B_1's and that of B_3 is the reverse of B_4's. Without loss of generality, we could assume that B_1 and B_3 have the same preference lists and that b_3 comes ahead of b_1 on their preference lists. Therefore, the partial preference lists of the men appear as follows.

B_1	$\cdots$	b_3	$\cdots$	b_1	$\cdots$	b_2	$\cdots$	
B_2			$\cdots$	b_2	$\cdots$	b_1	$\cdots$	b_3
B_3	$\cdots$	b_3	$\cdots$	b_4	$\cdots$			
B_4	$\cdots$	b_4	$\cdots$	b_3	$\cdots$			

The partial preference lists of the women are given below.

b_1	$\cdots$	B_2	$\cdots$	B_1	$\cdots$
b_2	$\cdots$	B_1	$\cdots$	B_2	$\cdots$
b_3	$\cdots$	B_4	$\cdots$	B_3	$\cdots$
b_4	$\cdots$	B_3	$\cdots$	B_4	$\cdots$

The pair (B_1, b_3) must not be a blocking pair to the male-optimal matching that pairs B_i to b_i. Since b_3 appears ahead of b_1 on B_1's preference list, B_1 should appear after B_3 on b_3's preference list. Therefore, the women's partial preference lists are as follows.

b_1	$\cdots$	B_2	$\cdots$	B_1	$\cdots$			
b_2	$\cdots$	B_1	$\cdots$	B_2	$\cdots$			
b_3			$\cdots$	B_4	$\cdots$	B_3	$\cdots$	B_1
b_4	$\cdots$	B_1	$\cdots$	B_3	$\cdots$	B_4	$\cdots$	

Since the female-optimal matching pairs (B_{2i-1}, b_{2i}) and (B_{2i}, b_{2i-1}), the pair (B_1, b_4) cannot be a blocking pair. Since B_1 appears ahead of B_3 on b_4's preference list, b_4 should appear after b_2 on B_1's preference list. So the men's partial preference lists are as follows.

B_1	$\cdots$	b_3	$\cdots$	b_1	$\cdots$	b_2	$\cdots$	b_4
B_2	$\cdots$	b_4	$\cdots$	b_2	$\cdots$	b_1	$\cdots$	b_3
B_3	$\cdots$	b_3	$\cdots$	b_1	$\cdots$	b_2	$\cdots$	b_4
B_4	$\cdots$	b_4	$\cdots$	b_2	$\cdots$	b_1	$\cdots$	b_3

Since the male-optimal matching pairs (B_i, b_i), we see that (B_2, b_4) cannot be a blocking pair. Since b_4 appears ahead of b_2 on B_2's preference list, B_2 should appear after B_4 on b_4's preference list. This gives us more information about the women's partial preference lists.

b_1	$\cdots$	B_2	$\cdots$	B_4	$\cdots$	B_3	$\cdots$	B_1
b_2	$\cdots$	B_1	$\cdots$	B_3	$\cdots$	B_4	$\cdots$	B_2
b_3	$\cdots$	B_2	$\cdots$	B_4	$\cdots$	B_3	$\cdots$	B_1
b_4	$\cdots$	B_1	$\cdots$	B_3	$\cdots$	B_4	$\cdots$	B_2

Comparing the preference lists for the men and the women, we observe that in the men's preference lists the women involved in one rotation are sandwiched by the women of the other rotation. A similar thing happens in the women's preference lists, except that the rotations reverse their roles here, i.e. if the women from rotation R sandwich the women from rotation R' in the men's preference lists, then the men from R' sandwich the men from R in the women's preference lists. Since this is true for the men and women in every pair of rotations, we could assume that all odd men have b_1 and b_2 as the innermost pair, enveloped by b_3 and b_4 and so on. In other words, the preference lists for the men are as follows.

$$\begin{array}{r|c}
B_1 & b_{2k-1}\ b_{2k-3}\ \cdots\ b_3\ b_1\ b_2\ b_4\ \cdots\ b_{2k-2}\ b_{2k} \\
B_2 & b_{2k}\ b_{2k-2}\ \cdots\ b_4\ b_2\ b_1\ b_3\ \cdots\ b_{2k-3}\ b_{2k-1} \\
B_3 & b_{2k-1}\ b_{2k-3}\ \cdots\ b_3\ b_1\ b_2\ b_4\ \cdots\ b_{2k-2}\ b_{2k} \\
\vdots & \vdots \qquad\qquad \vdots \\
B_{2k-2} & b_{2k}\ b_{2k-2}\ \cdots\ b_4\ b_2\ b_1\ b_3\ \cdots\ b_{2k-3}\ b_{2k-1} \\
B_{2k-1} & b_{2k-1}\ b_{2k-3}\ \cdots\ b_3\ b_1\ b_2\ b_4\ \cdots\ b_{2k-2}\ b_{2k} \\
B_{2k} & b_{2k}\ b_{2k-2}\ \cdots\ b_4\ b_2\ b_1\ b_3\ \cdots\ b_{2k-3}\ b_{2k-1}
\end{array}$$

This fixes the preference lists for the women and they are as follows.

$$\begin{array}{r|c}
b_1 & B_2\ B_4\ \cdots\ B_{2k-2}\ B_{2k}\ B_{2k-1}\ B_{2k-3}\ \cdots\ B_3\ B_1 \\
b_2 & B_1\ B_3\ \cdots\ B_{2k-3}\ B_{2k-1}\ B_{2k}\ B_{2k-2}\ \cdots\ B_4\ B_2 \\
b_3 & B_2\ B_4\ \cdots\ B_{2k-2}\ B_{2k}\ B_{2k-1}\ B_{2k-3}\ \cdots\ B_3\ B_1 \\
\vdots & \vdots \qquad\qquad \vdots \\
b_{2k-2} & B_1\ B_3\ \cdots\ B_{2k-3}\ B_{2k-1}\ B_{2k}\ B_{2k-2}\ \cdots\ B_4\ B_2 \\
b_{2k-1} & B_2\ B_4\ \cdots\ B_{2k-2}\ B_{2k}\ B_{2k-1}\ B_{2k-3}\ \cdots\ B_3\ B_1 \\
b_{2k} & B_1\ B_3\ \cdots\ B_{2k-3}\ B_{2k-1}\ B_{2k}\ B_{2k-2}\ \cdots\ B_4\ B_2
\end{array}$$

Suppose $1 \le i < k$. Recall that R_i is of the form $\{(B_{2i-1}, b_{2i-1}), (B_{2i}, b_{2i})\}$ and R_{i+1} is $\{(B_{2i+1}, b_{2i+1}), (B_{2i+2}, b_{2i+2})\}$. Now R_i moves b_{2i-1} from B_{2i-1}, which is below B_{2i+1} on its preference list to B_{2i}, which is above B_{2i+1} on its preference list. Hence the rotation R_i eliminates the pair (B_{2i+1}, b_{2i-1}). Also, R_{i+1} moves B_{2i+1} to b_{2i+2}, which is strictly worse for B_{2i+1} than b_{2i-1}. Thus, R_i explicitly precedes R_{i+1} (taking $M = B_{2i+1}$ and $w = b_{2i-1}$ in Definition 3.6). $\square$

6. Stable matchings in the *k*-Euclidean model

Having given our construction for the k-attribute setting, we now turn to the k-Euclidean model. We remind the reader that in this model every man, say A_i, is associated with two points in $\mathbb{R}^k$. One of the points, $\bar{A}_i$, denotes his *position* and the other, $\hat{A}_i$, denotes the position of his ideal partner. We refer to $\bar{A}_i$ as the *position point* of A_i and to $\hat{A}_i$ as the *preference point* of A_i. Similarly, each women has her own position and preference points. Each man ranks the women based on the Euclidean distance between his own preference point and the women's position points. In other words, if the distance between $\hat{A}_i$ and $\bar{b}$ is *less than* the distance between $\hat{A}_i$ and $\bar{c}$, then A_i prefers b over c (b appears higher in his preference list than c).

In this section we work in the 2-dimensional Euclidean model. Our goal here is to establish Theorem 3, which we repeat below.

Theorem 3. *#BIS $\equiv_{AP}$ #SM(k-Euclidean) when $k \ge 2$.*

Theorem 3 asserts that #*BIS* and #*SM*(k-Euclidean) are AP-interreducible for $k \ge 2$. Since the AP-reduction from #*SM*(k-Euclidean) to #*BIS* follows easily from known results (see Section 3.5), we now give an AP-reduction from #*BIS* to #*SM*(k-Euclidean).

As in Section 4, we will show how to take an instance G of #*BIS* and, in polynomial time, construct an instance I of #*SM*(k-Euclidean) so that the number of stable matchings of I is equal to the number of independent sets of G.

Let $G = (V_1 \cup V_2, E)$ be an instance of #*BIS* with $|E| = n$. We will construct a 2-Euclidean stable matching instance having $3n$ men and $3n$ women. Our construction will use the ρ-cycles and σ-cycles defined in Section 4.1.1 . To specify the stable matching instance, we now give position and preference points for the $3n$ men and women.

First, we position the $3n$ women $a_1, \ldots, a_n,\ b_1, \ldots, b_n, c_1, \ldots, c_n$ such that the b-women lie on the y-axis and the a-women and c-women lie on the x-axis. We represent woman w_i by $\bar{w}_i = (\bar{w}_i(x), \bar{w}_i(y))$, where $\bar{w}_i(x)$ and $\bar{w}_i(y)$ are her x- and y-coordinates. The coordinates of $\bar{a}_i, \bar{b}_i$ and $\bar{c}_i$ are $(\bar{a}_i(x), 0)$, $(0, \bar{b}_i(y))$ and $(\bar{c}_i(x), 0)$, respectively. We impose further restrictions on the coordinates of $\bar{a}_i, \bar{b}_i$, and $\bar{c}_i$.

$$\text{Let } \bar{b}_{\rho i}(y) = \bar{a}_i(x), \qquad \bar{c}_{\sigma^{-1} i}(x) = \bar{a}_i(x) - 0.7 \quad \text{for } 1 \le i \le n.$$

Fixing the x-coordinates of $\bar{a}_1, \ldots, \bar{a}_n$ therefore fixes the positions of all of the women. Suppose D_1 through D_l are the l cycles of σ of lengths p_1 through p_l, respectively. As before, let e_i be a representative element of cycle D_i. So $D_i = \{e_i, \sigma(e_i), \ldots, \sigma^{p_i-1}(e_i)\}$. Also as before, let $Rep(\sigma) = \{e_1, e_2 \ldots, e_l\}$ be the set of representative elements of the σ-cycles.

Let $W_i = \{a_x : x \in D_i\} \cup \{c_x : x \in D_i\}$. We set $p_0 = 0$. For woman $a_{\sigma^h e_j}$, where $e_j \in Rep(\sigma)$, and $0 \le h \le p_j - 1$, we set $\bar{a}_{\sigma^h e_j}(x) = \sum_{i=0}^{j-1} 2p_i + h + 1$. The position points of the women are as follows.

$$\text{For } e_j \in Rep(\sigma), \quad 0 \le h \le p_j - 1 \text{ let}$$
$$\bar{a}_{\sigma^h e_j} = \left(\sum_{i=0}^{j-1} 2p_i + h + 1,\ 0 \right),$$
$$\bar{b}_{\rho\sigma^h e_j} = \left(0,\ \sum_{i=0}^{j-1} 2p_i + h + 1 \right), \quad \text{and}$$
$$\bar{c}_{\sigma^{(h-1)} e_j} = \left(\sum_{i=0}^{j-1} 2p_i + h + 0.3,\ 0 \right).$$

Next we fix the locations in the x-y plane for the ideal partners of the men as follows, i.e. we specify the preference points for each man.

$$\text{Let } \epsilon = 1/100^n.$$
$$\text{For } e_j \in Rep(\sigma), \quad 0 \le h \le p_j - 1 \text{ let}$$
$$\hat{A}_{\sigma^h e_j} = \left(\sum_{i=0}^{j-1} 2p_i + h + 1,\ \sum_{i=0}^{j-1} 2p_i + h + 1 - \epsilon \right),$$
$$\hat{B}_{\sigma^h e_j} = \left(\sum_{i=0}^{j-1} 2p_i + h + 1,\ 1000^n \right), \quad \text{and}$$
$$\hat{C}_{\sigma^{(h-1)} e_j} = \left(\sum_{i=0}^{j-1} 2p_i + h + 0.6,\ 0 \right).$$

Having fixed the position of the women and the preference points for the men, we next fix the position of the men and the preference points of the women.

First, we position the $3n$ men $A_1, \ldots, A_n, B_1, \ldots, B_n, C_1, \ldots, C_n$ such that the C-men lie on the y-axis and the A-men and B-men lie on the x-axis. We represent man m_i by $\bar{M}_i = (\bar{M}_i(x), \bar{M}_i(y))$, where $\bar{M}_i(x)$ and $\bar{M}_i(y)$ are his x- and y-coordinates. The coordinates of $\bar{A}_i$, $\bar{B}_i$ and $\bar{C}_i$ are $(\bar{A}_i(x), 0)$, $(\bar{B}_i(x), 0)$ and $(0, \bar{C}_i(y))$, respectively. We impose further restrictions on the coordinates of $\bar{A}_i$, $\bar{B}_i$, and $\bar{C}_i$.

$$\text{Let } \bar{B}_i(x) = \bar{C}_i(y), \qquad \bar{A}_{\rho^{-1} i}(x) = \bar{B}_i(x) - 0.7 \quad \text{for } 1 \le i \le n.$$

Here again, fixing the x-coordinates of $\bar{B}_1, \ldots, \bar{B}_n$ therefore fixes the positions of all of the men. Suppose E_1 through E_k are the k cycles of ρ of lengths q_1 through q_k, respectively. As before, let f_i be a representative element of cycle E_i, which is the element of E_i with the smallest index. So $E_i = \{f_i, \rho(f_i), \ldots, \rho^{q_i-1}(f_i)\} = \{f_i, f_i + 1, \ldots, f_i + q_i - 1\}$. Also as before, let $Rep(\rho) = \{f_1, f_2 \ldots, f_k\}$ be the set of representative elements of the ρ-cycles.

Let $W_i = \{B_x : x \in E_i\} \cup \{A_x : x \in E_i\}$. We set $q_0 = 0$. For man $B_{\rho^h f_j}$, where $f_j \in Rep(\rho)$, and $0 \le h \le q_j - 1$, we set $\bar{B}_{\rho^h f_j}(x) = \sum_{i=0}^{j-1} 2q_i + h + 1$. The position points of the men are as follows.

$$\text{For } f_j \in Rep(\rho), \quad 0 \le h \le q_j - 1 \text{ let}$$
$$\bar{A}_{\rho^{h-1} f_j} = \left(\sum_{i=0}^{j-1} 2q_i + h + 0.3,\ 0 \right),$$
$$\bar{B}_{\rho^h f_j} = \left(\sum_{i=0}^{j-1} 2q_i + h + 1,\ 0 \right), \quad \text{and}$$
$$\bar{C}_{\rho^h f_j} = \left(0,\ \sum_{i=0}^{j-1} 2q_i + h + 1 \right).$$

Next we fix the locations in the x–y plane for the ideal partners of the women as follows, i.e. we specify the preference points for each woman.

Let $\epsilon = 1/100^n$.

For $f_j \in Rep(\rho), \quad 0 \le h \le q_j - 1$

$$\text{let } \hat{a}_{\rho^h f_j} = \left(\sum_{i=0}^{j-1} 2q_i + h + 1, \quad 1000^n \right),$$

$$\hat{b}_{\rho^h f_j} = \left(\sum_{i=0}^{j-1} 2q_i + h + 0.6, \; 0 \right), \quad \text{and}$$

$$\hat{c}_{\rho^h f_j} = \left(\sum_{i=0}^{j-1} 2q_i + h + 1, \; \sum_{i=0}^{j-1} 2q_i + h + 1 - \epsilon \right).$$

Having assigned position and preference points for both the men and the women, we construct the initial part of the preference lists of the men, starting with man $C_{\sigma^{(h-1)} e_j}$. We compare the distances of the women from $\hat{C}_{\sigma^{(h-1)} e_j}$ to produce the initial part of the preference list.

$e_j, e_m \in Rep(\sigma), \quad 0 \le f, h \le p_j - 1, \quad 0 \le g \le p_m - 1$

$$d^2(\hat{C}_{\sigma^{(h-1)} e_j}, \bar{b}_{\rho\sigma^g e_m}) = \left(\sum_{i=0}^{j-1} 2p_i + h + 0.6 - 0 \right)^2 + \left(0 - \sum_{i=0}^{m-1} 2p_i - g - 1 \right)^2 \ge 0.6^2 + 1^2 = 1.36$$

$$d^2(\hat{C}_{\sigma^{(h-1)} e_j}, \bar{c}_{\sigma^{(h-1)} e_j}) = \left(\sum_{i=0}^{j-1} 2p_i + h + 0.6 - \sum_{i=0}^{j-1} 2p_i - h - 0.3 \right)^2 + (0-0)^2 = 0.09 \quad \text{and}$$

$$d^2(\hat{C}_{\sigma^{(h-1)} e_j}, \bar{a}_{\sigma^h e_j}) = \left(\sum_{i=0}^{j-1} 2p_i + h + 0.6 - \sum_{i=0}^{j-1} 2p_i - h - 1 \right)^2 + (0-0)^2 = 0.16.$$

For $h \ne f$,

$$\begin{aligned} d^2(\hat{C}_{\sigma^{(h-1)} e_j}, \bar{c}_{\sigma^{(f-1)} e_j}) &= \left(\sum_{i=0}^{j-1} 2p_i + h + 0.6 - \sum_{i=0}^{j-1} 2p_i - f - 0.3 \right)^2 + (0-0)^2 \\ &\ge (|h-f| - 0.3)^2 \ge (1-0.3)^2 = 0.49 \quad \text{and} \end{aligned}$$

$$\begin{aligned} d^2(\hat{C}_{\sigma^{(h-1)} e_j}, \bar{a}_{\sigma^{(f)} e_j}) &= \left(\sum_{i=0}^{j-1} 2p_i + h + 0.6 - \sum_{i=0}^{j-1} 2p_i - f - 1 \right)^2 + (0-0)^2 \\ &\ge (|h-f| - 0.4)^2 \ge (1-0.4)^2 = 0.36. \end{aligned}$$

For $m > j$,

$$\begin{aligned} d^2(\hat{C}_{\sigma^{(h-1)} e_j}, \bar{c}_{\sigma^{(g-1)} e_m}) &= \left(\sum_{i=0}^{j-1} 2p_i + h + 0.6 - \sum_{i=0}^{m-1} 2p_i - g - 0.3 \right)^2 + (0-0)^2 \\ &\ge \left(\left| \sum_{i=j}^{m-1} 2p_i + g \right| - |h + 0.3| \right)^2 \\ &\ge (2p_j - (p_j - 1) - 0.3)^2 \ge (2 - 0.3)^2 = 2.89 \quad \text{and} \end{aligned}$$

$$\begin{aligned} d^2(\hat{C}_{\sigma^{(h-1)} e_j}, \bar{a}_{\sigma^g e_m}) &= \left(\sum_{i=0}^{j-1} 2p_i + h + 0.6 - \sum_{i=0}^{m-1} 2p_i - g - 1 \right)^2 + (0-0)^2 \\ &\ge \left(\left| \sum_{i=j}^{m-1} 2p_i + g + 0.4 \right| - |h| \right)^2 \\ &\ge (2p_j + 0.4 - (p_j - 1))^2 \ge 2.4^2 = 5.76. \end{aligned}$$

For $j > m$,

$$\begin{aligned} d^2(\hat{C}_{\sigma^{(h-1)}e_j}, \bar{c}_{\sigma^{(g-1)}e_m}) &= \left(\sum_{i=0}^{j-1} 2p_i + h + 0.6 - \sum_{i=0}^{m-1} 2p_i - g - 0.3\right)^2 + (0-0)^2 \\ &\geq \left(\left|\sum_{i=m}^{j-1} 2p_i + h + 0.3\right| - |g|\right)^2 \\ &\geq (2p_m + 0.3 - (p_m - 1))^2 \geq 2.3^2 = 5.29 \quad \text{and} \end{aligned}$$

$$\begin{aligned} d^2(\hat{C}_{\sigma^{(h-1)}e_j}, \bar{a}_{\sigma^g e_m}) &= \left(\sum_{i=0}^{j-1} 2p_i + h + 0.6 - \sum_{i=0}^{m-1} 2p_i - g - 1\right)^2 + (0-0)^2 \\ &\geq \left(\left|\sum_{i=m}^{j-1} 2p_i + h\right| - |g + 0.4|\right)^2 \\ &\geq (2p_m - (p_m - 1) - 0.4)^2 \geq (1.6)^2 = 2.56. \end{aligned}$$

From the above analysis, it follows that the preference list of $C_{\sigma^{(h-1)}e_j}$ starts with $c_{\sigma^{(h-1)}e_j} a_{\sigma^h e_j}$ for $1 \leq j \leq l,\ 0 \leq h \leq p_j - 1$ as in Section 4.1.3.

Now we carry out a similar analysis to determine the initial part of the preference list of $A_{\sigma^h e_j}$. We note that $\sum_{i=1}^{l} 2p_i = 2n$ and $2\epsilon \cdot 2n = \frac{4n}{100^n} \leq 0.04$. This implies that in the following analysis we could upper bound the term $2\epsilon \cdot (\sum_{i=1}^{j}(2p_i) + h + 1)$ by 0.04.

$$\epsilon = 1/100^n, \quad e_j, e_m \in Rep(\sigma), \quad 0 \leq f, h \leq p_j - 1, \quad 0 \leq g \leq p_m - 1$$

$$\begin{aligned} d^2(\hat{A}_{\sigma^h e_j}, \bar{a}_{\sigma^h e_j}) &= \left(\sum_{i=0}^{j-1} 2p_i + h + 1 - \sum_{i=0}^{j-1} 2p_i - h - 1\right)^2 + \left(\sum_{i=0}^{j-1} 2p_i + h + 1 - \epsilon - 0\right)^2 \\ &= \left(\sum_{i=0}^{j-1} 2p_i + h + 1 - \epsilon\right)^2 \end{aligned}$$

$$\begin{aligned} d^2(\hat{A}_{\sigma^h e_j}, \bar{b}_{\rho\sigma^h e_j}) &= \left(\sum_{i=0}^{j-1} 2p_i + h + 1 - 0\right)^2 + \left(\sum_{i=0}^{j-1} 2p_i + h + 1 - \epsilon - \sum_{i=0}^{j-1} 2p_i - h - 1\right)^2 \\ &= \left(\sum_{i=0}^{j-1} 2p_i + h + 1\right)^2 + \epsilon^2 \end{aligned}$$

$$\begin{aligned} d^2(\hat{A}_{\sigma^h e_j}, \bar{c}_{\sigma^{h-1}e_j}) &= \left(\sum_{i=0}^{j-1} 2p_i + h + 1 - \sum_{i=0}^{j-1} 2p_i - h - 0.3\right)^2 + \left(\sum_{i=0}^{j-1} 2p_i + h + 1 - \epsilon - 0\right)^2 \\ &= 0.7^2 + \left(\sum_{i=0}^{j-1} 2p_i + h + 1 - \epsilon\right)^2 \\ &= \left(\sum_{i=0}^{j-1} 2p_i + h + 1 - \epsilon\right)^2 + 0.49 \\ &\geq \left(\sum_{i=0}^{j-1} 2p_i + h + 1\right)^2 + \epsilon^2 + 0.45. \end{aligned}$$

For $h \neq f$,

$$\begin{aligned} d^2(\hat{A}_{\sigma^h e_j}, \bar{a}_{\sigma^f e_j}) &= \left(\sum_{i=0}^{j-1} 2p_i + h + 1 - \sum_{i=0}^{j-1} 2p_i - f - 1\right)^2 + \left(\sum_{i=0}^{j-1} 2p_i + h + 1 - \epsilon - 0\right)^2 \\ &= (h-f)^2 + \left(\sum_{i=0}^{j-1} 2p_i + h + 1 - \epsilon\right)^2 \end{aligned}$$

$$\geq \left(\sum_{i=0}^{j-1} 2p_i + h + 1 - \epsilon\right)^2 + 1$$
$$\geq \left(\sum_{i=0}^{j-1} 2p_i + h + 1\right)^2 + \epsilon^2 + 0.96$$

$$\begin{aligned} d^2(\hat{A}_{\sigma^h e_j}, \bar{b}_{\rho\sigma^f e_j}) &= \left(\sum_{i=0}^{j-1} 2p_i + h + 1 - 0\right)^2 + \left(\sum_{i=0}^{j-1} 2p_i + h + 1 - \epsilon - \sum_{i=0}^{j-1} 2p_i - f - 1\right)^2 \\ &= \left(\sum_{i=0}^{j-1} 2p_i + h + 1\right)^2 + (h - f - \epsilon)^2 \\ &\geq \left(\sum_{i=0}^{j-1} 2p_i + h + 1\right)^2 + (1 - \epsilon)^2 \\ &\geq \left(\sum_{i=0}^{j-1} 2p_i + h + 1\right)^2 + \epsilon^2 + 0.98 \end{aligned}$$

$$\begin{aligned} d^2(\hat{A}_{\sigma^h e_j}, \bar{c}_{\sigma^{f-1} e_j}) &= \left(\sum_{i=0}^{j-1} 2p_i + h + 1 - \sum_{i=0}^{j-1} 2p_i - f - 0.3\right)^2 + \left(\sum_{i=0}^{j-1} 2p_i + h + 1 - \epsilon - 0\right)^2 \\ &= (h - f - 0.7)^2 + \left(\sum_{i=0}^{j-1} 2p_i + h + 1 - \epsilon\right)^2 \\ &\geq \left(\sum_{i=0}^{j-1} 2p_i + h + 1 - \epsilon\right)^2 + 0.09 \\ &\geq \left(\sum_{i=0}^{j-1} 2p_i + h + 1\right)^2 + \epsilon^2 + 0.05 \end{aligned}$$

For $m > j$,

$$\begin{aligned} d^2(\hat{A}_{\sigma^h e_j}, \bar{a}_{\sigma^g e_m}) &= \left(\sum_{i=0}^{j-1} 2p_i + h + 1 - \sum_{i=0}^{m-1} 2p_i - g - 1\right)^2 + \left(\sum_{i=0}^{j-1} 2p_i + h + 1 - \epsilon - 0\right)^2 \\ &\geq \left(\left|\sum_{i=j}^{m-1} 2p_i + g\right| - |h|\right)^2 + \left(\sum_{i=0}^{j-1} 2p_i + h + 1 - \epsilon\right)^2 \\ &\geq (2p_j - (p_j - 1))^2 + \left(\sum_{i=0}^{j-1} 2p_i + h + 1 - \epsilon\right)^2 \\ &\geq \left(\sum_{i=0}^{j-1} 2p_i + h + 1 - \epsilon\right)^2 + 4 \\ &\geq \left(\sum_{i=0}^{j-1} 2p_i + h + 1\right)^2 + \epsilon^2 + 3.96 \end{aligned}$$

$$\begin{aligned} d^2(\hat{A}_{\sigma^h e_j}, \bar{b}_{\rho\sigma^g e_m}) &= \left(\sum_{i=0}^{j-1} 2p_i + h + 1 - 0\right)^2 + \left(\sum_{i=0}^{j-1} 2p_i + h + 1 - \epsilon - \sum_{i=0}^{m-1} 2p_i - g - 1\right)^2 \\ &\geq \left(\sum_{i=0}^{j-1} 2p_i + h + 1\right)^2 + \left(\left|\sum_{i=j}^{m-1} 2p_i + g + \epsilon\right| - |h|\right)^2 \\ &\geq \left(\sum_{i=0}^{j-1} 2p_i + h + 1\right)^2 + (2p_j + \epsilon - (p_j - 1))^2 \end{aligned}$$

$$\geq \left(\sum_{i=0}^{j-1} 2p_i + h + 1\right)^2 + (2+\epsilon)^2$$
$$\geq \left(\sum_{i=0}^{j-1} 2p_i + h + 1\right)^2 + \epsilon^2 + 4$$

$$\begin{aligned}
d^2(\hat{A}_{\sigma^h e_j}, \bar{c}_{\sigma^{g-1} e_m}) &= \left(\sum_{i=0}^{j-1} 2p_i + h + 1 - \sum_{i=0}^{m-1} 2p_i - g - 0.3\right)^2 + \left(\sum_{i=0}^{j-1} 2p_i + h + 1 - \epsilon - 0\right)^2 \\
&\geq \left(\left|\sum_{i=j}^{m-1} 2p_i + g\right| - |h + 0.7|\right)^2 + \left(\sum_{i=0}^{j-1} 2p_i + h + 1 - \epsilon\right)^2 \\
&\geq (2p_j - (p_j - 1) - 0.7)^2 + \left(\sum_{i=0}^{j-1} 2p_i + h + 1 - \epsilon\right)^2 \\
&\geq (2 - 0.7)^2 + \left(\sum_{i=0}^{j-1} 2p_i + h + 1 - \epsilon\right)^2 \\
&= \left(\sum_{i=0}^{j-1} 2p_i + h + 1 - \epsilon\right)^2 + 1.69 \\
&\geq \left(\sum_{i=0}^{j-1} 2p_i + h + 1\right)^2 + \epsilon^2 + 1.65.
\end{aligned}$$

For $j > m$,

$$\begin{aligned}
d^2(\hat{A}_{\sigma^h e_j}, \bar{a}_{\sigma^g e_m}) &= \left(\sum_{i=0}^{j-1} 2p_i + h + 1 - \sum_{i=0}^{m-1} 2p_i - g - 1\right)^2 + \left(\sum_{i=0}^{j-1} 2p_i + h + 1 - \epsilon - 0\right)^2 \\
&\geq \left(\left|\sum_{i=m}^{j-1} 2p_i + h\right| - |g|\right)^2 + \left(\sum_{i=0}^{j-1} 2p_i + h + 1 - \epsilon\right)^2 \\
&\geq (2p_m - (p_m - 1))^2 + \left(\sum_{i=0}^{j-1} 2p_i + h + 1 - \epsilon\right)^2 \\
&\geq \left(\sum_{i=0}^{j-1} 2p_i + h + 1 - \epsilon\right)^2 + 4 \\
&\geq \left(\sum_{i=0}^{j-1} 2p_i + h + 1\right)^2 + \epsilon^2 + 3.96
\end{aligned}$$

$$\begin{aligned}
d^2(\hat{A}_{\sigma^h e_j}, \bar{b}_{\rho\sigma^g e_m}) &= \left(\sum_{i=0}^{j-1} 2p_i + h + 1 - 0\right)^2 + \left(\sum_{i=0}^{j-1} 2p_i + h + 1 - \epsilon - \sum_{i=0}^{m-1} 2p_i - g - 1\right)^2 \\
&\geq \left(\sum_{i=0}^{j-1} 2p_i + h + 1\right)^2 + \left(\left|\sum_{i=m}^{j-1} 2p_i + h\right| - |g + \epsilon|\right)^2 \\
&\geq \left(\sum_{i=0}^{j-1} 2p_i + h + 1\right)^2 + (2p_m - (p_m - 1) - \epsilon)^2 \\
&\geq \left(\sum_{i=0}^{j-1} 2p_i + h + 1\right)^2 + (2 - \epsilon)^2 \\
&\geq \left(\sum_{i=0}^{j-1} 2p_i + h + 1\right)^2 + \epsilon^2 + 3.96
\end{aligned}$$

$$\begin{aligned} d^2(\hat{A}_{\sigma^h e_j}, \bar{c}_{\sigma^{g-1} e_m}) &= \left(\sum_{i=0}^{j-1} 2p_i + h + 1 - \sum_{i=0}^{m-1} 2p_i - g - 0.3\right)^2 + \left(\sum_{i=0}^{j-1} 2p_i + h + 1 - \epsilon - 0\right)^2 \\ &\geq \left(\left|\sum_{i=m}^{j-1} 2p_i + h + 0.7\right| - |g|\right)^2 + \left(\sum_{i=0}^{j-1} 2p_i + h + 1 - \epsilon\right)^2 \\ &\geq (2p_m - (p_m - 1) + 0.7)^2 + \left(\sum_{i=0}^{j-1} 2p_i + h + 1 - \epsilon\right)^2 \\ &\geq (2 + 0.7)^2 + \left(\sum_{i=0}^{j-1} 2p_i + h + 1 - \epsilon\right)^2 \\ &= \left(\sum_{i=0}^{j-1} 2p_i + h + 1 - \epsilon\right)^2 + 7.29 \\ &\geq \left(\sum_{i=0}^{j-1} 2p_i + h + 1\right)^2 + \epsilon^2 + 7.25. \end{aligned}$$

From the above analysis, it follows that the preference list of $A_{\sigma^h e_j}$ starts with $a_{\sigma^h e_j} b_{\rho\sigma^h e_j}$ for $1 \leq j \leq l,\ 0 \leq h \leq p_j - 1$ as in Section 4.1.3.

Last, we study the preference list of $B_{\sigma^h e_j}$. First we will show that the preference list of man $B_{\sigma^h e_j}$, where $1 \leq j \leq l,\ 0 \leq h \leq p_j - 1$, starts with

$$b_{\rho\sigma^{(p_l-1)} e_l} b_{\rho\sigma^{(p_l-2)} e_l} \cdots b_{\rho e_l} b_{\rho\sigma^{(p_{l-1}-1)} e_{l-1}} \cdots b_{\rho e_{l-1}} \cdots b_{\rho\sigma^{(p_1-1)} e_1} \cdots b_{\rho e_1}.$$

We obtain the above preference list by comparing distances between $\hat{B}_{\sigma^h e_j}$ and the positions of the women.

$$e_j, e_k \in Rep(\sigma), \qquad 0 \leq h \leq p_j - 1, \quad 0 \leq f \leq p_k - 1$$

$$\begin{aligned} d^2(\hat{B}_{\sigma^h e_j}, \bar{b}_{\rho\sigma^f e_k}) &= \left(\sum_{i=0}^{j-1} 2p_i + h + 1 - 0\right)^2 + \left(1000^n - \sum_{i=0}^{k-1} 2p_i - f - 1\right)^2 \\ &\leq \left(\sum_{i=0}^{j-1} 2p_i + h + 1\right)^2 + (1000^n - 1)^2 < (1000^n)^2 \end{aligned}$$

$$\begin{aligned} d^2(\hat{B}_{\sigma^h e_j}, \bar{a}_{\sigma^f e_k}) &= \left(\sum_{i=0}^{j-1} 2p_i + h + 1 - \sum_{i=0}^{k-1} 2p_i - f - 1\right)^2 + (1000^n - 0)^2 \\ &\geq (1000^n)^2 \end{aligned}$$

$$\begin{aligned} d^2(\hat{B}_{\sigma^h e_j}, \bar{c}_{\sigma^{f-1} e_k}) &= \left(\sum_{i=0}^{j-1} 2p_i + h + 1 - \sum_{i=0}^{k-1} 2p_i - f - 0.3\right)^2 + (1000^n - 0)^2 \\ &\geq (1000^n)^2. \end{aligned}$$

It immediately follows that the b-women are all closer to $\hat{B}_{\sigma^h e_j}$ than any of the a-women or c-women. Hence, the preference list of $B_{\sigma^h e_j}$ would start with all the b-women coming first. We also note that the b-women all have their x-component set to 0. Hence, $B_{\sigma^h e_j}$ would rank the b-women by measuring their distance from $\hat{B}_{\sigma^h e_j}$ in the y-component. We also note that $1000^n - 2n > 0$ for $n \geq 1$. Next we compare distances between $\hat{B}_{\sigma^h e_j}$ and the b-women only using the y-component. We will use the notation $d_y(\cdot, \cdot)$ to denote the distance in the y-component.

$$e_j, e_{k_1}, e_{k_2} \in Rep(\sigma), \quad 0 \leq h \leq p_j - 1, \quad 0 \leq g \leq p_{k_1} - 1, \quad 0 \leq f \leq p_{k_2} - 1.$$

For $k_1 = k_2$ and $g > f$, we have

$$\begin{aligned} d_y(\hat{B}_{\sigma^h e_j}, \bar{b}_{\rho\sigma^g e_{k_1}}) &= 1000^n - \sum_{i=0}^{k_1-1} 2p_i - g - 1 < 1000^n - \sum_{i=0}^{k_1-1} 2p_i - f - 1 \\ &= 1000^n - \sum_{i=0}^{k_2-1} 2p_i - f - 1 = d_y(\hat{B}_{\sigma^h e_j}, \bar{b}_{\rho\sigma^f e_{k_2}}). \end{aligned}$$

For $k_1 > k_2$, we have

$$\begin{aligned} d_y(\hat{B}_{\sigma^h e_j}, \bar{b}_{\rho\sigma^f e_{k_2}}) - d_y(\hat{B}_{\sigma^h e_j}, \bar{b}_{\rho\sigma^g e_{k_1}}) &= \left(1000^n - \sum_{i=0}^{k_2-1} 2p_i - f - 1\right) - \left(1000^n - \sum_{i=0}^{k_1-1} 2p_i - g - 1\right) \\ &= \sum_{i=0}^{k_1-1} 2p_i + g - \sum_{i=0}^{k_2-1} 2p_i - f = \sum_{i=k_2}^{k_1-1} 2p_i + g - f \\ &\geq 2p_{k_2} + g - (p_{k_2} - 1) = p_{k_2} + 1 + g > 0. \end{aligned}$$

From the above discussion, it follows that the preference list of $B_{\sigma^h e_j}$ starts with

$$b_{\rho\sigma^{(p_l-1)}e_l} b_{\rho\sigma^{(p_l-2)}e_l} \cdots b_{\rho e_l} b_{\rho\sigma^{(p_{l-1}-1)}e_{l-1}} \cdots b_{\rho e_{l-1}} \cdots b_{\rho\sigma^{(p_1-1)}e_1} \cdots b_{\rho e_1}.$$

Next we compare the distances of a-women and c-women from $\hat{B}_{\sigma^h e_j}$. As a-women and c-women all have their y-component set to 0, $B_{\sigma^h e_j}$ would rank the b-women by measuring their distance from $\hat{B}_{\sigma^h e_j}$ in the x-component. We will use the notation $d_x(\cdot, \cdot)$ to denote the distance in the x-component. We consider two cases (i) $h \neq p_i - 1,\ 1 \leq i \leq l$, (ii) $h = p_i - 1$, for some $i \in \{1, 2, \ldots, l\}$.

Case(i) $h \neq p_i - 1,\ 1 \leq i \leq l$: Now we compute and compare the distances between $\hat{B}_{\sigma^h e_j}$ and the a-women and the c-women.

$$e_j, e_k \in Rep(\sigma), \quad 0 \leq h \leq p_j - 2, \quad 0 \leq g \leq p_k - 1.$$

For $k = j$ and $g = h$, we have

$$\begin{aligned} d_x(\hat{B}_{\sigma^h e_j}, \bar{a}_{\sigma^g e_k}) &= d_x(\hat{B}_{\sigma^h e_j}, \bar{a}_{\sigma^h e_j}) \\ &= \left|\sum_{i=0}^{j-1} 2p_i + h + 1 - \sum_{i=0}^{j-1} 2p_i - h - 1\right| = 0 \quad \text{and} \\ d_x(\hat{B}_{\sigma^h e_j}, \bar{c}_{\sigma^g e_k}) &= d_x(\hat{B}_{\sigma^h e_j}, \bar{c}_{\sigma^h e_j}) \\ &= \left|\sum_{i=0}^{j-1} 2p_i + h + 1 - \sum_{i=0}^{j-1} 2p_i - (h+1) - 0.3\right| = 0.3. \end{aligned}$$

For $k = j$ and $g \notin \{h, p_j - 1\}$, we have

$$\begin{aligned} d_x(\hat{B}_{\sigma^h e_j}, \bar{a}_{\sigma^g e_k}) &= d_x(\hat{B}_{\sigma^h e_j}, \bar{a}_{\sigma^g e_j}) \\ &= \left|\sum_{i=0}^{j-1} 2p_i + h + 1 - \sum_{i=0}^{j-1} 2p_i - g - 1\right| = |h - g| \geq 1 \quad \text{and} \\ d_x(\hat{B}_{\sigma^h e_j}, \bar{c}_{\sigma^g e_k}) &= d_x(\hat{B}_{\sigma^h e_j}, \bar{c}_{\sigma^g e_j}) \\ &= \left|\sum_{i=0}^{j-1} 2p_i + h + 1 - \sum_{i=0}^{j-1} 2p_i - g - 1 - 0.3\right| \geq ||h - g| - 0.3| \geq 0.7. \end{aligned}$$

For $k = j$ and $g = p_j - 1$, we have

$$\begin{aligned} d_x(\hat{B}_{\sigma^h e_j}, \bar{a}_{\sigma^g e_k}) &= d_x(\hat{B}_{\sigma^h e_j}, \bar{a}_{\sigma^{p_j-1} e_j}) \\ &= \left|\sum_{i=0}^{j-1} 2p_i + h + 1 - \sum_{i=0}^{j-1} 2p_i - (p_j - 1) - 1\right| \\ &= |h - (p_j - 1)| \geq 1 \quad \text{(because } h \neq p_j - 1\text{)} \quad \text{and} \\ d_x(\hat{B}_{\sigma^h e_j}, \bar{c}_{\sigma^g e_k}) &= d_x(\hat{B}_{\sigma^h e_j}, \bar{c}_{\sigma^{p_j-1} e_j}) = d_x(\hat{B}_{\sigma^h e_j}, \bar{c}_{\sigma^{-1} e_j}) \\ &= \left|\sum_{i=0}^{j-1} 2p_i + h + 1 - \sum_{i=0}^{j-1} 2p_i - 0.3\right| \geq |h + 0.7| \geq 0.7. \end{aligned}$$

For $k > j$ we have

$$\begin{aligned}
d_x(\hat{B}_{\sigma^h e_j}, \bar{a}_{\sigma^g e_k}) &= \left|\sum_{i=0}^{j-1} 2p_i + h + 1 - \sum_{i=0}^{k-1} 2p_i - g - 1\right| \\
&= \left|\sum_{i=j}^{k-1} 2p_i + g - h\right| \\
&\geq |2p_j + g - (p_j - 1)| \geq |p_j + 1 + g| \geq 2 \quad \text{and} \\
d_x(\hat{B}_{\sigma^h e_j}, \bar{c}_{\sigma^{g-1} e_k}) &= \left|\sum_{i=0}^{j-1} 2p_i + h + 1 - \sum_{i=0}^{k-1} 2p_i - g - 0.3\right| \\
&= \left|\sum_{i=j}^{k-1} 2p_i + g - h - 0.7\right| \\
&\geq |2p_j + g - (p_j - 1) - 0.7| \geq |p_j + 1 + g - 0.7| \geq 1.3.
\end{aligned}$$

For $k < j$ we have

$$\begin{aligned}
d_x(\hat{B}_{\sigma^h e_j}, \bar{a}_{\sigma^g e_k}) &= \left|\sum_{i=0}^{j-1} 2p_i + h + 1 - \sum_{i=0}^{k-1} 2p_i - g - 1\right| = \left|\sum_{i=k}^{j-1} 2p_i + h - g\right| \\
&\geq |2p_k + h - (p_k - 1)| \geq |p_k + 1 + h| \geq 2 \quad \text{and} \\
d_x(\hat{B}_{\sigma^h e_j}, \bar{c}_{\sigma^{g-1} e_k}) &= \left|\sum_{i=0}^{j-1} 2p_i + h + 1 - \sum_{i=0}^{k-1} 2p_i - g - 0.3\right| = \left|\sum_{i=k}^{j-1} 2p_i + h + 0.7 - g\right| \\
&\geq |2p_k + h + 0.7 - (p_k - 1)| \geq |p_k + 1.7 + h| \geq 2.7.
\end{aligned}$$

It follows from the comparison that $B_{\sigma^h e_j}$ prefers $a_{\sigma^h e_j}$ over $c_{\sigma^h e_j}$ and $c_{\sigma^h e_j}$ over any other a-woman and the c-woman. Hence, the initial part of the preference list of $B_{\sigma^h e_j}$ reads

$$b_{\rho\sigma^{(p_l-1)} e_l} b_{\rho\sigma^{(p_l-2)} e_l} \cdots b_{\rho e_l} b_{\rho\sigma^{(p_{l-1}-1)} e_{l-1}} \cdots b_{\rho e_{l-1}} \cdots b_{\rho\sigma^{(p_1-1)} e_1} \cdots b_{\rho e_1} a_{\sigma^h e_j} c_{\sigma^h e_j}.$$

Case(ii) $h = p_i - 1$, for some $i \in \{1, 2, \ldots, l\}$: Now we compare the distances between $\hat{B}_{\sigma^h e_j}$ and the a-women and the c-women.

$$e_j, e_k \in Rep(\sigma), \quad h = p_j - 1, \quad 0 \leq g \leq p_k - 1.$$

For $k = j$ and $g = h$, we have

$$\begin{aligned}
d_x(\hat{B}_{\sigma^h e_j}, \bar{a}_{\sigma^g e_k}) &= d_x(\hat{B}_{\sigma^{p_j-1} e_j}, \bar{a}_{\sigma^{p_j-1} e_j}) \\
&= \left|\sum_{i=0}^{j-1} 2p_i + h + 1 - \sum_{i=0}^{j-1} 2p_i - h - 1\right| = 0 \quad \text{and} \\
d_x(\hat{B}_{\sigma^h e_j}, \bar{c}_{\sigma^g e_k}) &= d_x(\hat{B}_{\sigma^{p_j-1} e_j}, \bar{c}_{\sigma^{p_j-1} e_j}) = d_x(\hat{B}_{\sigma^{p_j-1} e_j}, \bar{c}_{\sigma^{-1} e_j}) \\
&= \left|\sum_{i=0}^{j-1} 2p_i + p_j - \sum_{i=0}^{j-1} 2p_i - 0.3\right| = p_j - 0.3.
\end{aligned}$$

For $k > j$ we have

$$\begin{aligned}
d_x(\hat{B}_{\sigma^{p_j-1} e_j}, \bar{a}_{\sigma^g e_k}) &= \left|\sum_{i=0}^{j-1} 2p_i + p_j - \sum_{i=0}^{k-1} 2p_i - g - 1\right| \\
&= \left|\sum_{i=j}^{k-1} 2p_i + g + 1 - p_j\right| \geq |2p_j + g + 1 - p_j| \\
&\geq |p_j + 1 + g| \geq p_j + 1 \quad \text{and}
\end{aligned}$$

$$\begin{aligned}
d_x(\hat{B}_{\sigma^h e_j}, \bar{c}_{\sigma^{g-1} e_k}) &= \left|\sum_{i=0}^{j-1} 2p_i + p_j - \sum_{i=0}^{k-1} 2p_i - g - 0.3\right| \\
&= \left|\sum_{i=j}^{k-1} 2p_i + g + 0.3 - p_j\right| \geq |2p_j + g + 0.3 - p_j| \\
&\geq |p_j + 0.3 + g| \geq p_j + 0.3.
\end{aligned}$$

For $k < j$ we have

$$\begin{aligned}
d_x(\hat{B}_{\sigma^{p_j-1} e_j}, \bar{a}_{\sigma^g e_k}) &= \left|\sum_{i=0}^{j-1} 2p_i + p_j - \sum_{i=0}^{k-1} 2p_i - g - 1\right| \\
&= \left|\sum_{i=k}^{j-1} 2p_i + p_j - g - 1\right| \geq |2p_k + p_j - p_k| \\
&\geq |p_k + p_j| \geq p_j + 1 \quad \text{and}
\end{aligned}$$

$$\begin{aligned}
d_x(\hat{B}_{\sigma^h e_j}, \bar{c}_{\sigma^{g-1} e_k}) &= \left|\sum_{i=0}^{j-1} 2p_i + p_j - \sum_{i=0}^{k-1} 2p_i - g - 0.3\right| \\
&= \left|\sum_{i=k}^{j-1} 2p_i + p_j - g - 0.3\right| \geq |2p_k + p_j - (p_k - 1) - 0.3| \\
&\geq |p_k + p_j + 0.7| \geq p_j + 1.7.
\end{aligned}$$

From the above inequalities, it follows that $B_{\sigma^h e_j}$ prefers $a_{\sigma^{p_j-1} e_j}$ over $c_{\sigma^{-1} e_j}$, and $c_{\sigma^{-1} e_j}$ over a-women and c-women whose subscript belongs to σ cycles different from that of $c_{\sigma^{-1} e_j}$'s. Now we compute and compare distances from $\hat{B}_{\sigma^h e_j}$ to all the a-women and c-women whose subscript is on the same σ cycle as $a_{\sigma^{p_j-1} e_j}$'s and $c_{\sigma^{-1} e_j}$'s.
For $0 \leq g \leq p_j - 2$ we have

$$\begin{aligned}
d_x(\hat{B}_{\sigma^{p_j-1} e_j}, \bar{c}_{\sigma^g e_j}) - d_x(\hat{B}_{\sigma^{p_j-1} e_j}, \bar{a}_{\sigma^g e_j}) &= \left|\sum_{i=0}^{j-1} 2p_i + p_j - \sum_{i=0}^{j-1} 2p_i - g - 1 - 0.3\right| - \left|\sum_{i=0}^{j-1} 2p_i + p_j - \sum_{i=0}^{j-1} 2p_i - g - 1\right| \\
&= p_j - g - 1.3 - (p_j - g - 1) < 0
\end{aligned}$$

and for $0 \leq g \leq p_j - 1$ we have

$$\begin{aligned}
d_x(\hat{B}_{\sigma^{p_j-1} e_j}, \bar{a}_{\sigma^g e_j}) - d_x(\hat{B}_{\sigma^{p_j-1} e_j}, \bar{c}_{\sigma^{g-1} e_j}) &= \left|\sum_{i=0}^{j-1} 2p_i + p_j - \sum_{i=0}^{j-1} 2p_i - g - 1\right| - \left|\sum_{i=0}^{j-1} 2p_i + p_j - \sum_{i=0}^{j-1} 2p_i - g - 0.3\right| \\
&= p_j - g - 1 - (p_j - g - 0.3) < 0.
\end{aligned}$$

From the above comparisons, it follows that $B_{\sigma^{p_j-1} e_j}$ prefers $a_{\sigma^{p_j-1} e_j}$ over $c_{\sigma^{p_j-2} e_j}$, $c_{\sigma^g e_j}$ over $a_{\sigma^g e_j}$ and $a_{\sigma^g e_j}$ over $c_{\sigma^{g-1} e_j}$ for $0 \leq g \leq p_j - 2$. Stringing these preferences together, we obtain a portion of $B_{\sigma^{p_j-1} e_j}$'s preference list, which appears as

$$a_{\sigma^{p_j-1} e_j} c_{\sigma^{p_j-2} e_j} a_{\sigma^{p_j-2} e_j} c_{\sigma^{p_j-3} e_j} \cdots c_{\sigma e_j} a_{\sigma e_j} c_{e_j} a_{e_j} c_{\sigma^{-1} e_j} (= c_{\sigma^{p_j-1} e_j}).$$

We remind the reader that $B_{\sigma^{p_j-1} e_j}$'s preference list has all the b-women appearing at the front appended by the above list of a-women and c-women. Hence, the initial part of $B_{\sigma^{p_j-1} e_j}$'s preference list is

$$b_{\rho\sigma^{(p_l-1)} e_l} \cdots b_{\rho e_l} b_{\rho\sigma^{(p_{l-1}-1)} e_{l-1}} \cdots b_{\rho e_{l-1}} \cdots b_{\rho\sigma^{(p_1-1)} e_1} \cdots b_{\rho e_1}$$

$$a_{\sigma^{(p_j-1)} e_j} c_{\sigma^{(p_j-2)} e_j} a_{\sigma^{(p_j-2)} e_j} \cdots c_{e_j} a_{e_j} c_{\sigma^{-1} e_j} (= c_{\sigma^{(p_j-1)} e_j}).$$

The initial part of the preference lists of men $A_{\sigma^s e_i}$, $C_{\sigma^{s-1} e_i}$ and $B_{\sigma^s e_i}$ are as follows.

$$
\begin{aligned}
& e_i \in Rep(\sigma), \\
A_{\sigma^m e_i} \quad &: \quad a_{\sigma^m e_i} b_{\rho\sigma^m e_i}, \quad 0 \le m \le p_i - 1 \\
C_{\sigma^{(m-1)} e_i} \quad &: \quad c_{\sigma^{(m-1)} e_i} a_{\sigma^m e_i}, \quad 0 \le m \le p_i - 1 \\
B_{\sigma^m e_i} \quad &: \quad b_{\rho\sigma^{(p_l-1)} e_l} \cdots b_{\rho e_l} b_{\rho\sigma^{(p_{l-1}-1)} e_{l-1}} \cdots b_{\rho e_{l-1}} \cdots \\
& \qquad b_{\rho\sigma^{(p_1-1)} e_1} \cdots b_{\rho e_1} a_{\sigma^m e_i} c_{\sigma^m e_i}, \quad 0 \le m \le p_i - 2 \\
B_{\sigma^{(p_i-1)} e_i} \quad &: \quad b_{\rho\sigma^{(p_l-1)} e_l} \cdots b_{\rho e_l} b_{\rho\sigma^{(p_{l-1}-1)} e_{l-1}} \cdots b_{\rho e_{l-1}} \cdots b_{\rho\sigma^{(p_1-1)} e_1} \cdots b_{\rho e_1} a_{\sigma^{(p_i-1)}} c_{\sigma^{(p_i-2)} e_i} \\
& \qquad a_{\sigma^{(p_i-2)} e_i} \cdots a_{\sigma e_i} c_{e_i} a_{e_i} c_{\sigma^{(p_i-1)} e_i}.
\end{aligned}
$$

Note that these are exactly the same as those in (4) for any appropriate value of the permutation τ. In a similar manner, we can obtain preference lists for the women. The preference lists for the women are as follows.

$$
\begin{aligned}
& f_i \in Rep(\rho), \\
b_{\rho^m f_i} \quad &: \quad A_{\rho^{(m-1)} f_i} B_{\rho^m f_i}, \quad 0 \le m \le q_i - 1 \\
c_{\rho^m f_i} \quad &: \quad B_{\rho^m f_i} C_{\rho^m f_i}, \quad 0 \le m \le q_i - 1 \\
a_{\rho^m f_i} \quad &: \quad C_{\rho^{(q_k-1)} f_k} \cdots C_{f_k} C_{\rho^{(q_{k-1}-1)} f_{k-1}} \cdots C_{f_{k-1}} \cdots \\
& \qquad C_{\rho^{(q_1-1)} f_1} \cdots C_{f_1} B_{\rho^m f_i} A_{\rho^m f_i}, \quad 0 \le m \le q_i - 2 \\
a_{\rho^{(q_i-1)} f_i} \quad &: \quad C_{\rho^{(q_k-1)} f_k} \cdots C_{f_k} C_{\rho^{(q_{k-1}-1)} f_{k-1}} \cdots C_{f_{k-1}} B_{\rho^{(q_i-1)} f_i} A_{\rho^{(q_i-2)} f_i} \\
& \qquad B_{\rho^{(q_i-2)} f_i} \cdots B_{\rho f_i} A_{f_i} B_{f_i} A_{\rho^{(q_i-1)} f_i}.
\end{aligned}
$$

Now note that, by the construction of the ρ cycles, which go in order from 1 to n, the list

$$C_{\rho^{(q_k-1)} f_k} \cdots C_{f_k} C_{\rho^{(q_{k-1}-1)} f_{k-1}} \cdots C_{f_{k-1}} \cdots C_{\rho^{(q_1-1)} f_1} \cdots C_{f_1}$$

is identically $C_n \cdots C_1$. Thus, the preference lists for the women are identical to those given in (3). Thus, the rest of the proof is exactly the same as in the 3-attribute case, starting from the introduction of the men's lists (4) in Section 4.2.

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